

Unit 4

Trigonometry

Introduction

Trigonometry is about the properties of triangles. A typical problem in trigonometry requires you to calculate one of the side lengths or angles of a triangle from other information about the triangle. For instance, you might need to calculate the side length c of the triangle in Figure 1. Problems of this type arise in a wealth of scientific subjects, from astronomy to civil engineering, and also in everyday life. Suppose, for example, that you want to estimate the width of a river. Trigonometry provides you with mathematical tools to approach this task, as you will see later.

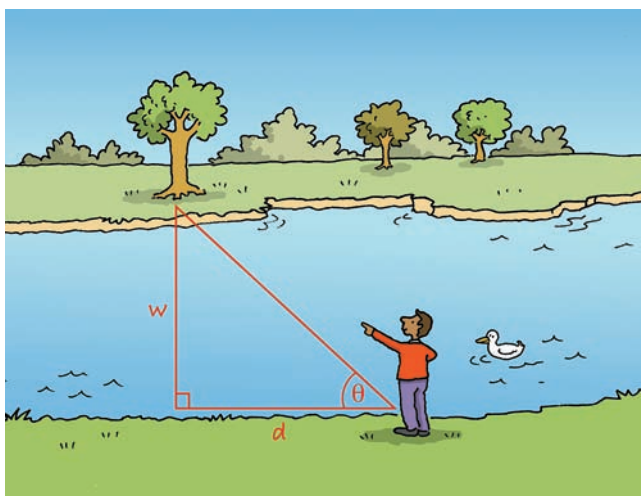


Figure 2 What is the width of the river?

Trigonometry goes far beyond triangles, though. All the angles of a triangle measure between 0° and 180° , but trigonometry also deals with larger angles, such as 300° or 500° , and even with negative angles, such as -100° . These deeper aspects of trigonometry have applications in electrical engineering and acoustics, among other scientific disciplines, as well as in more sophisticated branches of mathematics, such as *Fourier series*.

The term ‘trigonometry’ is derived from the Greek words *trigonon* for triangle and *metron* for measure. It first appeared in the book *Trigonometriæ*, by Bartholomeo Pitiscus, published in 1595.

Section 1 of this unit is about trigonometry in right-angled triangles (triangles with a right angle). You’ll revise the trigonometric ratios sine, cosine and tangent, and use them to find angles and side lengths.

The angles of a right-angled triangle all measure between 0° and 90° . In Section 2, you’ll learn the meanings of the sine, cosine and tangent of *any* angle (even negative angles). In this setting, sine, cosine and tangent are described as *trigonometric functions*: they are functions because each of them is a rule that takes an input number (the size of an angle) and

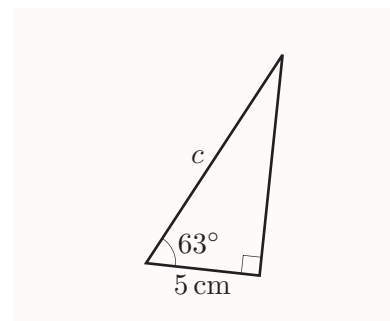


Figure 1 What is the length of the side c ? You’ll learn how to solve problems like this in Section 1



The frontispiece from Bartholomeo Pitiscus’s *Trigonometriæ* (1612 edition)

produces an output number. In this section you'll also meet the graphs of the trigonometric functions, and learn about inverse trigonometric functions.

Section 3 returns to triangles, but considers those that don't necessarily have a right angle. You'll see how to use two important trigonometric rules, the sine rule and the cosine rule, to find angles and side lengths in such triangles.

Finally, in Section 4, you'll learn about many useful relationships between trigonometric functions, which are conveniently expressed as *trigonometric identities*. You'll need to use these identities when you study calculus, in Units 6 to 8, and complex numbers, in Unit 12.

Throughout the unit you'll practise working with angles measured in *radians*, rather than degrees, in preparation for topics that you'll study later in the module.

1 Right-angled triangles

In this section you'll revise the trigonometry of right-angled triangles. We begin with a reminder about *radians*.

1.1 Radians and degrees

Radians are units for measuring angles, which can be used as an alternative to degrees. They're used extensively in this module, and in many higher-level mathematics modules, as they're more convenient to work with in many contexts. For example, many mathematical formulas are simpler when angles are measured in radians rather than degrees. You'll see two such formulas in this subsection. When you study calculus and complex numbers later in the module, you'll use radians rather than degrees whenever you work with angles.

To understand what a radian is, consider any circle. An unbroken piece of its circumference is called an **arc**. For example, Figure 3 shows an arc of a circle. The centre of the circle is labelled O . The angle at the centre is said to be **subtended** by the arc.

Suppose that the circle has radius r , and you draw an arc also of length r on its circumference, as shown in Figure 4. Then the angle subtended by the arc is 1 radian. If the arc were of length $2r$ then the angle would be 2 radians, and so on. The definition of a radian is summarised in the following box.

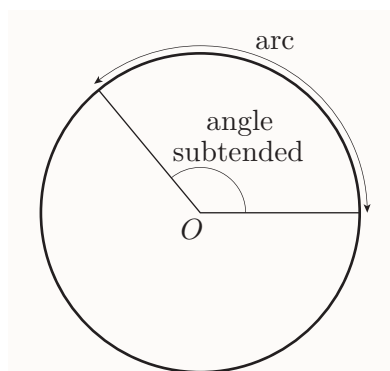


Figure 3 The angle subtended by an arc

Radians

One **radian** is the angle subtended at the centre of a circle by an arc that has the same length as the radius.

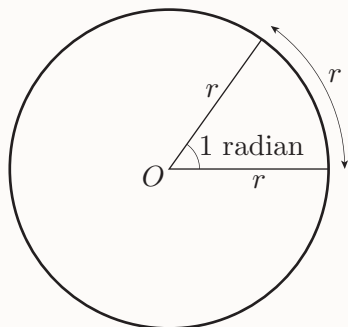


Figure 4 An angle of one radian

From this definition, you can find the number of radians in a full turn – that is, a turn of 360° (the symbol $^\circ$ means degrees). The circumference of a circle of radius r is $2\pi r$, and each arc of length r subtends an angle of 1 radian. So the number of radians in a full turn is

$$\frac{2\pi r}{r} = 2\pi.$$

In other words, 360° is the same angle as 2π radians (about 6.28 radians).

$$2\pi \text{ radians} = 360^\circ$$

This gives

$$1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}.$$

Since

$$\frac{180}{\pi} = 57.295\dots,$$

one radian is approximately 57° .

In historical terms, the term ‘radian’ – a contraction of ‘radial angle’ – is relatively new. It appeared in print for the first time in 1873, in examination papers set by Professor James Thomson (brother of Lord Kelvin) at Queen’s College, Belfast. Who actually coined the term radian is a matter of some debate. As a unit of measurement, but without the name, it was in use much earlier, appearing in the work of Roger Cotes (1682–1716), a close colleague of Isaac Newton.

Because a full turn is 2π radians, the number of radians in a simple fraction of a full turn can be conveniently expressed in terms of π . For instance, a quarter turn (90°) is $\pi/2$ radians, and a third of a full turn (120°) is $2\pi/3$ radians. Similarly, a twelfth of a full turn (30°) is $2\pi/12 = \pi/6$ radians. It is usual to leave these numbers of radians in this form, rather than finding decimal approximations. Reasoning in this way, we can assemble Table 1.

Table 1 A conversion table for common angles

Angle in degrees	Angle in radians
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
360°	2π

As with all fractions, you should usually write fractions of π in ‘vertical’ fraction notation, rather than using a slash symbol. For example, it is better to write $\frac{\pi}{2}$ rather than $\pi/2$ when you are doing an algebraic manipulation with this expression, in order to distinguish the numerator and the denominator clearly. However, we often use the slash symbol notation where a fraction is part of a line of typed text.

The fraction $\pi/2$ is usually read as ‘pi by two’, though you can also say ‘pi over two’. Other fractions of π are read in a similar way.

Since

$$1 \text{ radian} = \frac{180^\circ}{\pi},$$

the factor $180/\pi$ can be used to convert an angle measured in radians into degrees, and vice versa, as detailed below.

Converting between degrees and radians

$$\text{number of radians} = \frac{\pi}{180} \times \text{number of degrees}$$

$$\text{number of degrees} = \frac{180}{\pi} \times \text{number of radians}$$

Here are some examples of these types of conversions.

Example 1 *Converting between degrees and radians*

(a) Convert 120° to radians.

(b) Convert $5\pi/3$ radians to degrees.

Solution

(a)  Apply the degrees-to-radians conversion formula. 

$$120^\circ = \left(\frac{\pi}{180} \times 120 \right) \text{ radians} = \frac{2\pi}{3} \text{ radians}$$

(b)  Apply the radians-to-degrees conversion formula. 

$$\frac{5\pi}{3} \text{ radians} = \left(\frac{180}{\pi} \times \frac{5\pi}{3} \right)^\circ = 300^\circ$$

Here are some similar questions for you to try.

Activity 1 *Converting between degrees and radians*

(a) Convert the following angles from degrees to radians.

(i) 1° (ii) 345°

(b) Convert the following angles from radians to degrees.

(i) $\frac{3\pi}{5}$ radians (ii) $\frac{7\pi}{4}$ radians

As mentioned at the beginning of the subsection, some mathematical formulas are simpler when angles are measured in radians rather than degrees. An example is the formula for the length of an arc on the circumference of a circle of radius r , in terms of the subtended angle of size θ (in radians). (The symbol θ is the lower-case form of the Greek letter theta, which is pronounced ‘thee-ta’. It is often used to represent angles.)

You know that

an arc that subtends an angle of 1 radian has length r .

Hence

an arc that subtends an angle of θ radians has length $r\theta$.

Such an arc is shown in Figure 5.

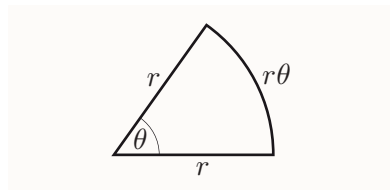


Figure 5 The arc length is $r\theta$

Length of an arc of a circle

$$\text{arc length} = r\theta,$$

where r is the radius of the circle and θ is the angle subtended by the arc, measured in radians.

For example, the formula tells you that an arc on a circle of radius 21 cm that subtends an angle of $\pi/7$ radians has the following length (in cm):

$$21 \times \frac{\pi}{7} = 3\pi = 9.42 \text{ (to 3 s.f.)}.$$

If the angle θ is measured in degrees rather than radians, then the formula for the length of an arc becomes

$$\text{arc length} = \frac{\pi}{180} \times r\theta = \frac{\pi r\theta}{180}.$$

This is a more cumbersome formula, which illustrates why radians are favoured over degrees in mathematical formulas.

Another formula that's simpler when radians are used rather than degrees is the formula for the area of a sector. A **sector** of a circle is the 'slice' of the circle lying between two radii, as shown in Figure 5. (Strictly speaking, a circle is just a curve, and a sector is really a slice of the *inside* of the circle. However, it is convenient to allow the word 'circle' to mean either a circle or its inside. With this convention, you can refer to the 'area of a circle', rather than the 'area enclosed by a circle'.)

Let's now obtain a formula for the area of a sector. Consider a circle of radius r . Its full area is πr^2 . Half of the circle is a sector of angle π radians (a half turn), which has area $\frac{1}{2}\pi r^2$. That is,

$$\text{an angle of } \pi \text{ radians gives a sector of area } \frac{1}{2}\pi r^2.$$

Hence

$$\text{an angle of 1 radian gives a sector of area } \frac{1}{2}r^2,$$

so

$$\text{an angle of } \theta \text{ radians gives a sector of area } \frac{1}{2}r^2\theta.$$

Area of a sector of a circle

$$\text{area of sector} = \frac{1}{2}r^2\theta,$$

where r is the radius of the circle and θ is the angle of the sector, measured in radians.

In the next activity, you can practise using the formulas for the length of an arc and the area of a sector.

Activity 2 *Using the arc length and sector area formulas*

The classic computer game character shown in Figure 6 is constructed by removing a sector of angle $\pi/3$ radians from a circle of radius 1 cm.

- The shape is itself a sector – of what angle?
- What is the total perimeter of the shape? Give your answer in cm, to one decimal place.
- What is the area of the shape? Give your answer in cm^2 , to one decimal place.

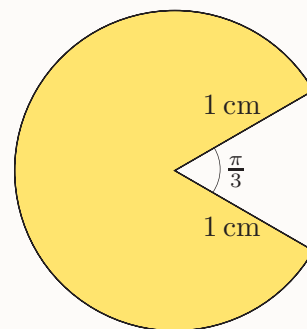


Figure 6 A circle with a sector of angle $\pi/3$ removed

Although radians usually give simpler mathematical formulas than degrees, there are many circumstances where degrees are the more appropriate choice of unit. This is because it can be more convenient to work with the large whole number 360 (for the number of degrees in a full turn) rather than the irrational number 2π (the number of radians in a full turn). Degrees tend to be used in practical situations – for example, by carpenters who want to measure the angles at the corners of pieces of wood.

In this module you will use both radians and degrees, and it is crucial that you always know what the current choice of unit is, especially when entering angle values into your calculator. Most calculators have an angle mode option for switching between degrees and radians. If your calculator is set to the wrong mode, then you will obtain incorrect results.

As you have seen, you must include the symbol $^\circ$ after the size of an angle measured in degrees. So far, we have indicated that the size of an angle is measured in radians by using the word ‘radians’. For example, you have seen angles written as $\pi/2$ radians and $2\pi/3$ radians. In fact, it is usual to omit the word ‘radians’, and write these angles simply as $\pi/2$ and $2\pi/3$.

From now on, if the size of an angle is stated without the symbol $^\circ$, then the angle is measured in radians.

To help you be aware of the choice of unit, you should remember that angles in degrees are usually written as whole numbers or decimals, such as 37° and 156.4° , whereas angles in radians are usually written as fractions of π .

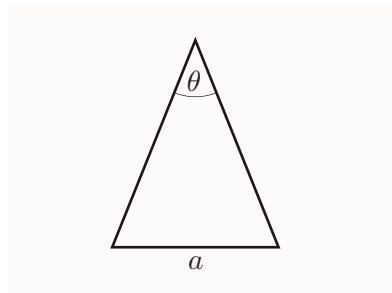


Figure 7 A triangle

Some useful conventions

Before you go on to Subsection 1.2, in which you'll start your revision of trigonometry, here are some standard conventions that we'll use in this subsection and throughout the rest of the unit.

First, we often use the same letter to denote either a line segment or its length. For example, we might say that the side a of the triangle in Figure 7 is horizontal, or we might write that a is 4 cm. We treat angles in a similar way. For example, we might say that the angle θ in Figure 7 is opposite the side a , or we might write that $\theta = \pi/4$.

Second, we often state the lengths of line segments, such as the sides of triangles, with no units. For example, we might say that one side of a triangle has length 5. When this is done, you should assume that the units are 'abstract' units, in the sense that they could represent any units that are used consistently. If you need to refer to them, then you can call them simply 'units'.

Finally, if you are asked to solve an abstract trigonometric problem, where no units of length are given, then (unless the question specifies otherwise) you should express your answers as exact values, rather than decimal approximations, whenever it is possible to do this reasonably concisely. For example, you might express an answer as a fraction or a surd, or in terms of π . If you are asked to solve a practical trigonometric problem, then you should normally give your answers as decimal approximations.

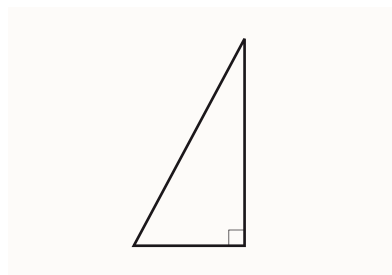


Figure 8 A right-angled triangle

1.2 Sine, cosine and tangent

In this section you'll begin your revision of trigonometry by considering the relationships between angles and side lengths in *right-angled triangles*. A **right-angled triangle** is simply one that contains a right angle, that is, an angle of $\pi/2$ (90°), as illustrated in Figure 8.

Each of the other two angles in a right-angled triangle is an **acute** angle, that is, an angle greater than 0 (0°) but less than $\pi/2$ (90°). This is because the three angles of a triangle add up to π (180°).

You already know a relationship between the three side lengths of a right-angled triangle, namely Pythagoras' theorem, which is repeated below. Remember that the side opposite the right angle in a right-angled triangle is called the **hypotenuse**, and it is always the longest side.

Pythagoras' theorem

For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

So if you know the lengths of two sides of a right-angled triangle, then you can use Pythagoras' theorem to work out the length of the third side. You saw some examples of this in Section 6 of Unit 1.

Sometimes, however, you may want to solve a problem that involves angles as well as side lengths in right-angled triangles. For example, you might have a triangle like the one in Figure 9(a), and you want to know the length of the side a . Or you might have a triangle like the one in Figure 9(b), and you want to know the size of the angle θ .

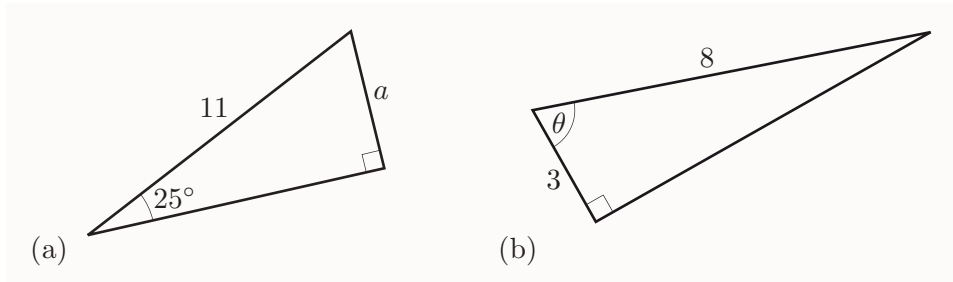


Figure 9 An unknown side a and an unknown angle θ

Problems like these can be solved by using the *sine*, *cosine* and *tangent* of an angle. These are numbers associated with the angle. All angles have a sine, cosine and tangent (except that some exceptional angles have no tangent), but in this section you'll consider the sine, cosine and tangent only of *acute* angles, as that's all you need to solve problems involving right-angled triangles. You'll learn about the sine, cosine and tangent of other angles later in the unit.

Here's how the sine, cosine and tangent of an acute angle θ are defined. Consider any right-angled triangle with θ as one of its acute angles, and label the sides of the triangle as follows: *hyp* for the *hypotenuse*, *opp* for the side *opposite* the angle θ , and *adj* for the side (other than the hypotenuse) *adjacent* to the angle θ . This is illustrated in Figure 10.

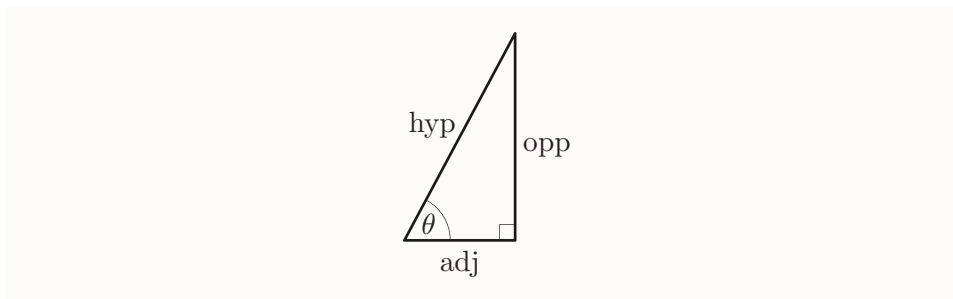


Figure 10 The opposite side, adjacent side and hypotenuse for an angle θ
The sine, cosine and tangent of θ are defined as follows.

Sine, cosine and tangent

The **sine** of the angle θ is

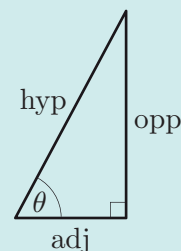
$$\sin \theta = \frac{\text{opp}}{\text{hyp}}.$$

The **cosine** of the angle θ is

$$\cos \theta = \frac{\text{adj}}{\text{hyp}}.$$

The **tangent** of the angle θ is

$$\tan \theta = \frac{\text{opp}}{\text{adj}}.$$



Sine, cosine and tangent are often referred to as sin, cos and tan, pronounced 'sine', 'coz' and 'tan', respectively.

A popular method for remembering the definitions above is to take the initial letters from

Sine = Opp/Hyp, Cosine = Adj/Hyp, Tangent = Opp/Adj, to make the acronym SOH CAH TOA. This acronym is read to rhyme with *Krakatoa* (a famous volcano in Indonesia).

SOH CAH TOA

Another mnemonic used for these letters is 'Silly Old Harry, Chased A Horse, Through Our Attic'.



Let's try out the definitions of sine, cosine and tangent with the angle θ of the triangle shown in Figure 11. We obtain

$$\sin \theta = \frac{4}{5}, \quad \cos \theta = \frac{3}{5} \quad \text{and} \quad \tan \theta = \frac{4}{3}.$$

The other acute angle in the triangle in Figure 11 (the smallest angle) does not have the same values of sine, cosine and tangent as the angle θ . This is because the side opposite this other angle is not the side opposite the angle θ , and similarly the side adjacent to this angle is not the side adjacent to the angle θ .

You always get the same answers for the sine, cosine and tangent of a particular angle θ , no matter which right-angled triangle with θ as one of its acute angles you consider. To see why, first notice that all the different right-angled triangles with θ as one of their acute angles have the same three angles. This is because the sum of the three angles in a triangle is π (180°), so the angle other than θ and the right angle is given by

$$\pi - \frac{\pi}{2} - \theta = \frac{\pi}{2} - \theta.$$

It follows that all the right-angled triangles with θ as one of their angles are scaled-up or scaled-down (and possibly flipped over) versions of each other – in other words, they are all **similar**. For example, Figure 12 shows three different right-angled triangles with a particular acute angle θ . You can see that they are all essentially the same shape.

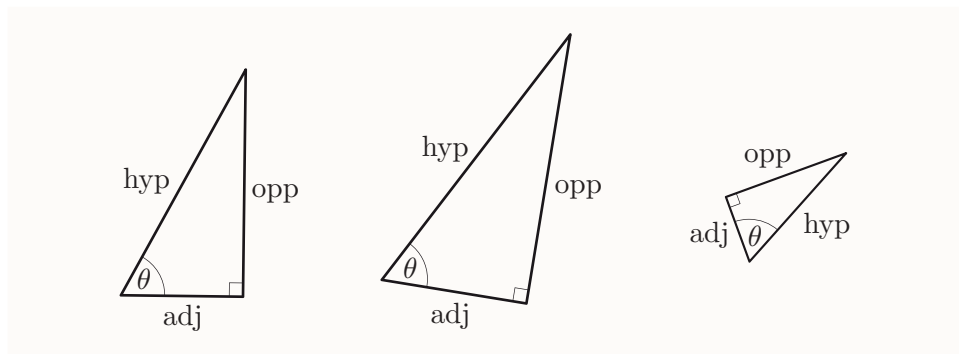


Figure 12 Similar triangles

Since all the right-angled triangles with a particular angle θ are scaled-up or scaled-down (and possibly flipped over) versions of each other, they all have their sides in the same proportions, and hence they all give the same values for $\sin \theta$, $\cos \theta$ and $\tan \theta$.

To help you see this, consider scaling the triangle shown in Figure 11 by a factor a , so that the new side lengths are $3a$, $4a$ and $5a$, as shown in Figure 13. Then

$$\sin \theta = \frac{4a}{5a} = \frac{4}{5}, \quad \cos \theta = \frac{3a}{5a} = \frac{3}{5} \quad \text{and} \quad \tan \theta = \frac{4a}{3a} = \frac{4}{3}.$$

These values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ are the same as those obtained from the original triangle in Figure 11.

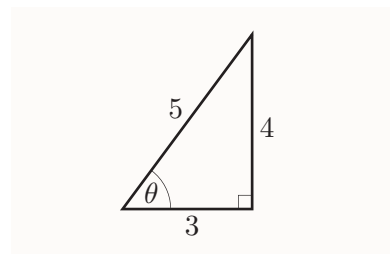


Figure 11 A right-angled triangle with side lengths 3, 4 and 5

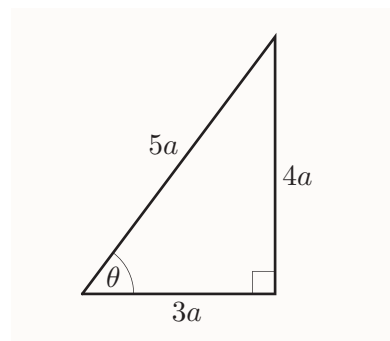


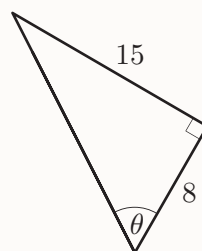
Figure 13 A right-angled triangle with side lengths $3a$, $4a$, and $5a$

The sine, cosine and tangent of an angle are known as **trigonometric ratios**, because each of them is defined as the ratio of one side of a right-angled triangle to another. (Remember that if a and b are two quantities, such as the lengths of sides of a triangle, then the **ratio** of a to b is a/b .)

You can write $\sin \theta$ as $\sin(\theta)$ if you prefer, and a similar comment applies to $\cos$ and $\tan$.

Activity 3 Finding trigonometric ratios

Consider the angle θ in the triangle below.



- Use Pythagoras' theorem to find the length of the hypotenuse.
- Find $\sin \theta$, $\cos \theta$ and $\tan \theta$.

If you know the sine, cosine or tangent of a particular acute angle, then you can find the other two trigonometric ratios for the angle by sketching a suitable right-angled triangle.

Activity 4 Finding trigonometric ratios from another trigonometric ratio

Suppose that θ is an acute angle and $\sin \theta = \frac{5}{13}$. Sketch a right-angled triangle with θ as one of its acute angles, and, by using Pythagoras' theorem, determine $\cos \theta$ and $\tan \theta$.

You can obtain the values of the sine, cosine and tangent of an angle using your calculator. You have to make sure that it is set to radians or degrees, as appropriate, or you will obtain incorrect answers.

Activity 5 Using a calculator to find trigonometric ratios

Use your calculator to find the values of $\sin 62^\circ$, $\cos 62^\circ$ and $\tan 62^\circ$, to three significant figures.

In fact the angle θ in Activity 3 is about 62° , so the answers that you obtained in Activity 5 should be close to those that you obtained in Activity 3. (They won't be exactly the same, as the angle isn't exactly 62° .)

The Greek astronomer and mathematician Hipparchus of Rhodes (c. 190–120 BC) is widely recognised as the founder of trigonometry. He created the first table of trigonometric ratios, to enable him to calculate astronomical data. His work was used by another astronomer and mathematician, Ptolemy (c. AD 90–168), whose texts had considerable impact on the development of trigonometry and astronomy.



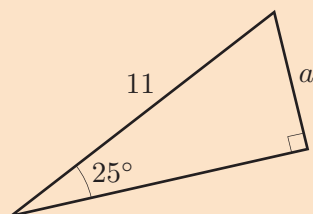
1.3 Finding unknown side lengths in right-angled triangles

If you know one of the acute angles and one of the side lengths in a right-angled triangle, then you can use trigonometric ratios to find either of the other two side lengths. The next example illustrates this.

An image from the frontispiece of William Cunningham's *Cosmographical Glass* (1559). The same frontispiece was used for Henry Billingsley's *Euclid* (1570), the first edition of Euclid's *Elements* in English

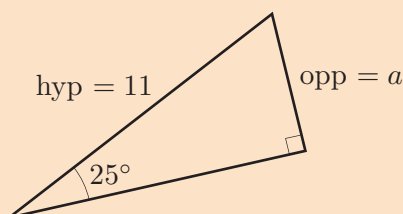
Example 2 Finding an unknown side length

Find the side length a in the triangle below, to two significant figures.



Solution

Label the sides of the triangle in relation to the given angle.



The two labelled sides are opp and hyp, so use the equation $\sin 25^\circ = \text{opp/hyp}$.

$$\sin 25^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{a}{11}$$

Rearrange this equation to find a .

$$a = 11 \sin 25^\circ$$

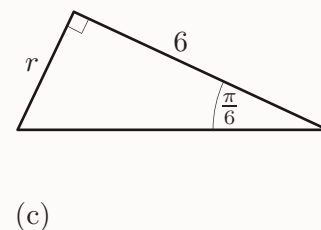
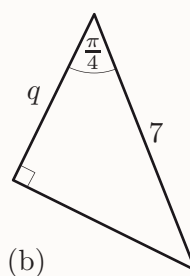
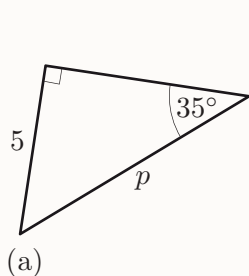
Find $11 \sin 25^\circ$ using your calculator.

$$a = 4.648 \dots = 4.6 \quad (\text{to 2 s.f.}).$$

You can practise calculating side lengths using sine, cosine and tangent in the next activity. The angle in part (a) is given in degrees, whereas the angles in parts (b) and (c) are given in radians, so make sure that your calculator is set to the appropriate unit each time.

Activity 6 Finding unknown side lengths

Calculate the side lengths p , q and r in the triangles below, to two significant figures.



Let's now use trigonometry to solve the problem of how to estimate the width of a river, which was mentioned in the introduction to the unit. First you find a landmark on the opposite bank of the river. Let's suppose you choose a tree. You move to the point on your side of the river directly opposite the tree. You then walk alongside the river to a point further along. At this point you look back towards your starting position, and then towards the tree, and measure the angle between these two lines of sight. This is shown in Figure 14: the width of the river is w , the distance that you walk is d and the angle is θ .

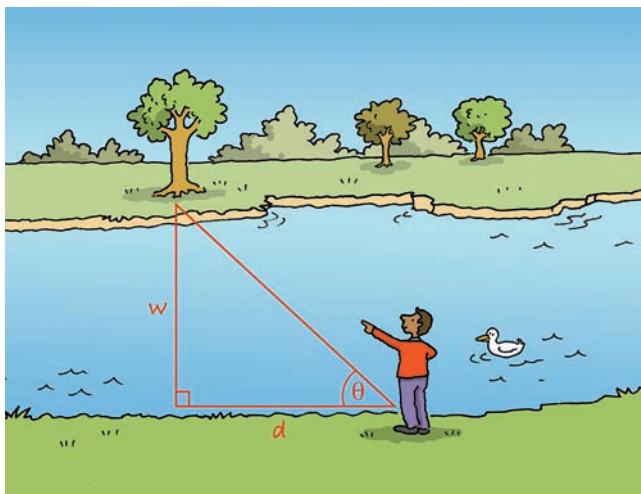


Figure 14 A right-angled triangle can be used to estimate the width of the river

From the diagram,

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{w}{d},$$

so

$$w = d \tan \theta.$$

You can now calculate w from d and θ ; the better your measurements of d and θ are, the better your estimate of w is.

Activity 7 *Estimating the width of a river*

Suppose that you carry out the procedure described above, and the distance that you walk is 50 m. If the angle between your two lines of sight is 57° , how wide is the river? Give your answer to the nearest metre.

When you use the method described above, it's best to walk a reasonable distance – perhaps roughly as far as the river is wide. If you walk only a short distance, then you're likely to obtain an inaccurate estimate, because then the angle θ will be close to 90° , and small errors in the measurement of an angle close to 90° result in large errors in the value of its tangent. You can see this from the graph of tangent, which is given in Subsection 2.3.

The next activity involves a similar application of trigonometry.

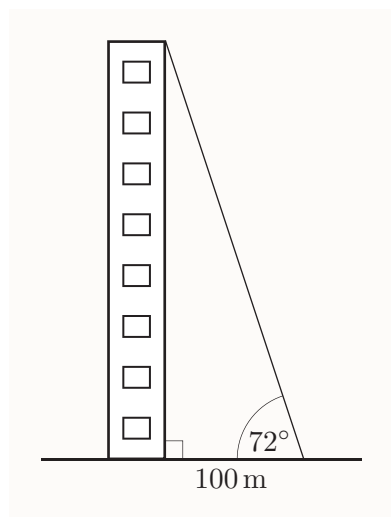


Figure 15 A tall building

Activity 8 *Estimating the height of a building*

Suppose that you stand 100 m from a tall building and measure the angle between the horizontal and the top of the building as 72° , as in Figure 15. How high is the building? Give your answer to the nearest metre.

1.4 Finding unknown angles in right-angled triangles

Let's now turn to the problem of finding an angle in a right-angled triangle when you know two side lengths. Take, for example, the triangle in Figure 16.

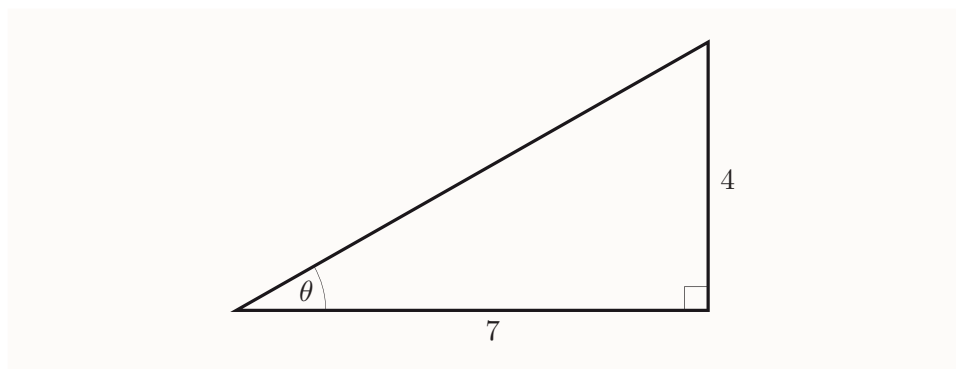


Figure 16 A right-angled triangle with two known side lengths

The side opposite the angle θ has length 4, and the side adjacent to the angle θ has length 7. Therefore

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{7}.$$

If you know the tangent of an acute angle (or the sine or cosine of the angle), then you can use your calculator to find the angle. The notation used for the acute angle whose tangent is the number x is

$$\tan^{-1} x.$$

Similarly, the notations used for the acute angle whose sine is x and the acute angle whose cosine is x are

$$\sin^{-1} x \quad \text{and} \quad \cos^{-1} x,$$

respectively.

Note that you may choose to write either $\tan^{-1} x$ or $\tan^{-1}(x)$, whichever seems clearer, just as you may choose to write either $\tan \theta$ or $\tan(\theta)$. A similar comment applies to sine and cosine.

The angles $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ are called the **inverse sine**, **inverse cosine** and **inverse tangent** of the number x , respectively. You can use your calculator to find them. For the triangle in Figure 16, $\tan \theta = 4/7$, so you can work out using your calculator that

$$\theta = \tan^{-1} \left(\frac{4}{7} \right) = 29.74 \dots^\circ = 29.7^\circ \quad (\text{to 3 s.f.}).$$

The notation used for inverse sine, inverse cosine and inverse tangent is essentially the notation for inverse functions that you met in Unit 3. This is because inverse sine, inverse cosine and inverse tangent are types of inverse functions. You'll learn more about this in Subsection 2.4.

A common error when working with inverse sine, inverse cosine or inverse tangent is to confuse $\tan^{-1} x$ with $(\tan x)^{-1}$, for example. Remember that

$$(\tan x)^{-1} = \frac{1}{\tan x},$$

and this is very different from $\tan^{-1} x$. A similar warning applies to $\sin^{-1} x$ and $\cos^{-1} x$.

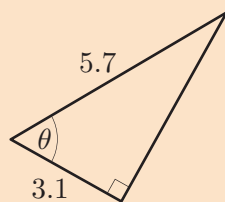
The inverse sine, inverse cosine and inverse tangent of a number x are also called the **arcsine**, **arccosine** and **arctangent** of x , with the following alternative notations:

$$\arcsin(x) = \sin^{-1} x, \quad \arccos(x) = \cos^{-1} x, \quad \arctan(x) = \tan^{-1} x.$$

Here's another example of finding an unknown angle in a right-angled triangle.

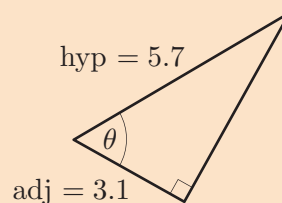
Example 3 *Finding an unknown angle*

Find the angle θ in the triangle below, to the nearest degree.



Solution

Label the sides of the triangle in relation to the given angle.



The two labelled sides are adj and hyp, so use the equation $\cos \theta = \text{adj}/\text{hyp}$.

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{3.1}{5.7}$$

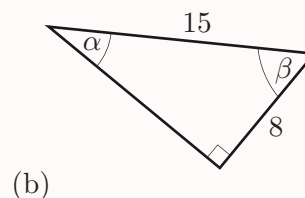
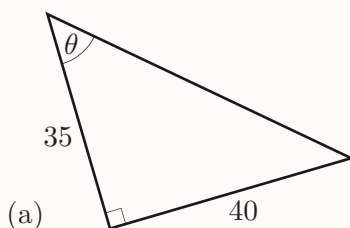
Use inverse cosine on your calculator to find θ .

$$\theta = \cos^{-1} \left(\frac{3.1}{5.7} \right) = 57.053 \dots^\circ = 57^\circ \quad (\text{to the nearest degree})$$

In the next activity, you can practise finding unknown angles in right-angled triangles. Notice that two of the angles in this activity are labelled α and β : these are the lower-case forms of the Greek letters alpha and beta, which are pronounced ‘al-fa’ and ‘bee-ta’. Angles are often represented by Greek letters. The full Greek alphabet is given in the *Handbook*.

Activity 9 Finding unknown angles

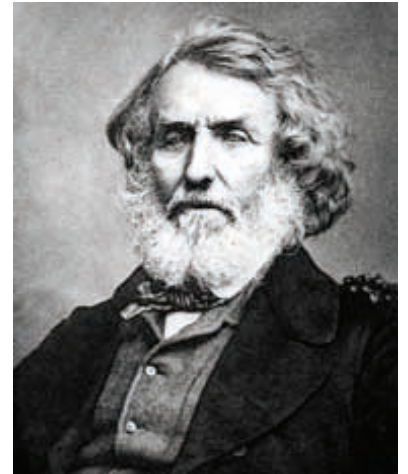
Find the angles θ , α and β in the triangles below, to the nearest degree.



It is important to remember that the strategies that you’ve met so far for finding unknown side lengths and angles are valid only in *right-angled* triangles. In Section 3 you’ll learn how to find unknown side lengths and angles in other triangles.

The Great Trigonometric Survey

The Great Trigonometric Survey was a survey of the geography of India, carried out in the nineteenth century. Trigonometry was fundamental to the survey, and was used to calculate measurements such as the heights of mountains in the Himalayas, including Mount Everest and K2. The superintendent of the latter parts of the survey was Colonel Sir George Everest, after whom Mount Everest is named.



George Everest (1790–1866)

1.5 Trigonometric ratios of special angles

You can work out the trigonometric ratios for the special angles $\pi/6$, $\pi/4$ and $\pi/3$ (that is, 30° , 45° and 60° , respectively) without using a calculator. These values are useful to know. Later in the unit you'll learn how to calculate sine, cosine and tangent for other special angles.

Let's begin with the angle $\pi/3$ (60°). This angle is one of the angles of an *equilateral triangle*, that is, a triangle in which all angles are equal. This is because $\pi/3$ (60°) is a third of the total angle sum π (180°) of a triangle.

Figure 17 shows an equilateral triangle that is split into two right-angled triangles by a vertical line. The equilateral triangle has sides of length 2, and the vertical line divides the base of the triangle into two equal parts, each of length 1. (Choosing the equilateral triangle to have sides of length 2 makes the calculations easier.)

By Pythagoras' theorem, the length of the vertical line is

$$\sqrt{2^2 - 1^2} = \sqrt{3},$$

as marked in the diagram. You can now apply the formulas

$$\sin \frac{\pi}{3} = \frac{\text{opp}}{\text{hyp}}, \quad \cos \frac{\pi}{3} = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \tan \frac{\pi}{3} = \frac{\text{opp}}{\text{adj}}$$

to the right-angled triangle on the left-hand side of the equilateral triangle in Figure 17. This gives

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \tan \frac{\pi}{3} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

Remember that you can write $\sin \frac{\pi}{3}$ as $\sin\left(\frac{\pi}{3}\right)$ or $\sin(\pi/3)$ if you prefer, and a similar comment applies for cos and tan.

You can find the trigonometric ratios for the angle $\pi/6$ (30°) by using the same diagram, Figure 17. This is because the other acute angle in the left-hand triangle in Figure 17 is $\pi/6$, as marked in Figure 18, since it is half of the top angle $\pi/3$ of the equilateral triangle. This gives

$$\sin \frac{\pi}{6} = \frac{1}{2}, \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

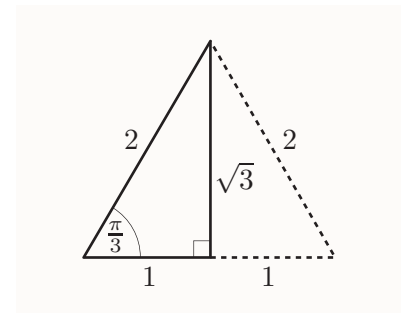


Figure 17 An equilateral triangle split into two right-angled triangles

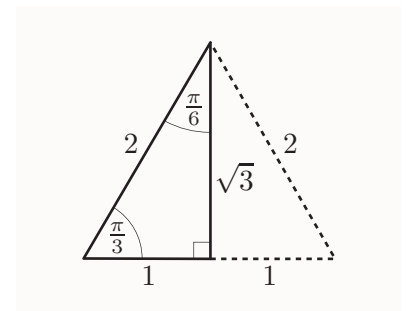


Figure 18 The right-angled triangle on the left has acute angles $\pi/3$ and $\pi/6$

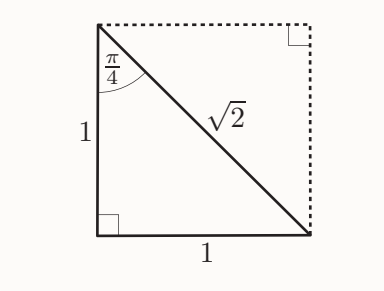


Figure 19 A right-angled triangle with two acute angles of size $\pi/4$

Let’s now consider the angle $\pi/4$ (45°). To find the trigonometric ratios for this angle, you can use the right-angled triangle shown in Figure 19.

It is created by splitting a square down one of its diagonals. Because each angle of a square is a right angle, $\pi/2$, each of the acute angles of the triangle is half of a right angle, $\pi/4$. By Pythagoras’ theorem, the length of the diagonal line is

$$\sqrt{1^2 + 1^2} = \sqrt{2},$$

as marked in the diagram. You can now apply the formulas

$$\sin \frac{\pi}{4} = \frac{\text{opp}}{\text{hyp}}, \quad \cos \frac{\pi}{4} = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \tan \frac{\pi}{4} = \frac{\text{opp}}{\text{adj}}$$

to obtain

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \tan \frac{\pi}{4} = 1.$$

The trigonometric ratios for the angles $\pi/6$, $\pi/4$ and $\pi/3$ are listed in Table 2, which will be referred to as the *special angles table*. This table is also given in the *Handbook*. It’s useful to learn these trigonometric ratios, or to learn how to work them out quickly by sketching the triangles in Figures 18 and 19, as they come up frequently.

Table 2 Sine, cosine and tangent of special angles

θ in radians	θ in degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Note that the surd $1/\sqrt{2}$, which appears in Table 2, can also be written as $\sqrt{2}/2$. Your calculator may display it like this. This alternative form is obtained by multiplying both the numerator and the denominator of the original fraction by $\sqrt{2}$, to rationalise the denominator. Similarly the surd $1/\sqrt{3}$ can be written as $\sqrt{3}/3$.

You can also use Table 2 to find some values of inverse sine, inverse cosine and inverse tangent. For example, from the first row you can see that

$$\cos^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{6}.$$

Notice that the values of sine and cosine in Table 2 are all less than 1. This is because the longest side of any right-angled triangle is the

hypotenuse, so opp and adj are both smaller than hyp. It follows that, for any acute angle θ , the values of

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \text{and} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}}$$

are both less than 1. However, as you can see from Table 2, the tangent of an acute angle can be less than 1, equal to 1, or greater than 1.

Spherical geometry

The angles of a triangle always add up to π (180°) ... or do they? In *plane* geometry (which is the only geometry that you'll meet in this module) they certainly do, but in *spherical* geometry the sum of the angles of a triangle is always greater than π ! The Earth, which is approximately spherical, provides a useful model for thinking about spherical geometry. The triangle shown in Figure 20, with one vertex at the North Pole and the other two vertices on the equator – one in Ecuador and the other in Gabon – has three right angles!

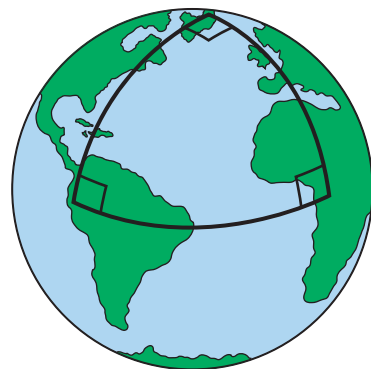


Figure 20 A spherical triangle with three right angles

1.6 Trigonometric identities

In Unit 1 you learned that an identity is an equation that is satisfied by all possible values of its variables. A **trigonometric identity** is an identity that involves one or more trigonometric ratios. In this subsection you'll meet several important trigonometric identities.

For our first trigonometric identity, consider any acute angle θ , and choose a right-angled triangle that has θ as one of its acute angles, and whose hypotenuse has length 1, as shown in Figure 21. You can always obtain such a triangle by scaling up or down any right-angled triangle that has θ as one of its acute angles. (The hypotenuse is chosen to have length 1 to simplify the algebra later on.)

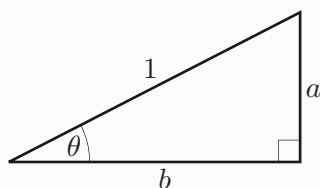


Figure 21 A right-angled triangle with hypotenuse of length 1 and other sides of lengths a and b

For this triangle,

$$\sin \theta = \frac{a}{1} = a, \quad \cos \theta = \frac{b}{1} = b \quad \text{and} \quad \tan \theta = \frac{a}{b}.$$

It follows that

$$\frac{\sin \theta}{\cos \theta} = \frac{a}{b} = \tan \theta,$$

which is our first trigonometric identity.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

This equation is an identity because it is true for *any* acute angle θ .

To obtain our second identity, let's apply Pythagoras's theorem to the right-angled triangle in Figure 21. This gives

$$a^2 + b^2 = 1^2 = 1.$$

Since $\sin \theta = a$ and $\cos \theta = b$, it follows that

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

It is conventional to write $(\sin \theta)^2$ and $(\cos \theta)^2$ as $\sin^2 \theta$ and $\cos^2 \theta$, respectively, so the identity above is usually written as follows.

$$\sin^2 \theta + \cos^2 \theta = 1$$

The notation $\sin^2 \theta$ for $(\sin \theta)^2$ is convenient as it avoids some brackets. Similarly we usually write $\sin^3 \theta$ for $(\sin \theta)^3$, and so on. However, remember that $\sin^{-1} x$ does *not* mean $(\sin x)^{-1}$, which is a way of writing $1/\sin x$. Instead $\sin^{-1} x$ is the inverse sine of x . This is an awkward clash of notation that you just need to become used to.

You can use the two trigonometric identities above to deduce the values of all the trigonometric ratios for a particular acute angle if you know the value of one of these ratios. This is an alternative to sketching a suitable right-angled triangle, as you were asked to do in Activity 4, and you're asked to try it in the following activity.

Activity 10 *Finding trigonometric ratios from another trigonometric ratio by using trigonometric identities*

Suppose that the acute angle θ is such that $\cos \theta = \frac{3}{7}$. Use the identities above to find the exact values of $\sin \theta$ and $\tan \theta$.

The next two trigonometric identities involve both the acute angles in a right-angled triangle. The sum of these two angles is $\pi/2$, so if one acute angle is θ , then the other is $(\pi/2) - \theta$, as shown in Figure 22.

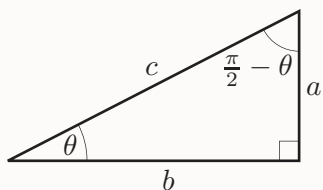


Figure 22 A right-angled triangle with acute angles θ and $(\pi/2) - \theta$

You can see from Figure 22 that

$$\sin \theta = \frac{a}{c} \quad \text{and} \quad \cos \theta = \frac{b}{c},$$

and also

$$\sin \left(\frac{\pi}{2} - \theta \right) = \frac{b}{c} \quad \text{and} \quad \cos \left(\frac{\pi}{2} - \theta \right) = \frac{a}{c}.$$

This gives us two more identities.

$$\sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta$$

$$\cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta$$

These identities tell you that if you have two acute angles that add up to $\pi/2$ (90°), then the sine of one is the cosine of the other, and vice versa. This explains the repeated values that you can see in the sine and cosine columns of the special angles table (Table 2 on page 22). For example, if $\theta = \pi/6$ then $(\pi/2) - \theta = \pi/3$, so the first of the two identities gives

$$\sin \frac{\pi}{3} = \cos \frac{\pi}{6},$$

which is confirmed by Table 2.

You're asked to prove an identity involving $\tan((\pi/2) - \theta)$ in the next activity.

Activity 11 *Proving a trigonometric identity*

Using Figure 22, show that, for every acute angle θ ,

$$\tan \left(\frac{\pi}{2} - \theta \right) = \frac{1}{\tan \theta}.$$

2 Trigonometric functions

In this section you'll learn what's meant by the sine, cosine and tangent of angles of any size, and you'll see that sine, cosine and tangent are functions.

2.1 Angles of any size

In Section 1 you worked with acute angles and right angles. However, you can also have:

- **obtuse** angles (those between 90° and 180° , that is, between $\pi/2$ and π)
- **straight** angles (those of 180° , that is, π)
- **reflex** angles (those between 180° and 360° , that is, between π and 2π).

These five types of angle are illustrated in Figure 23.

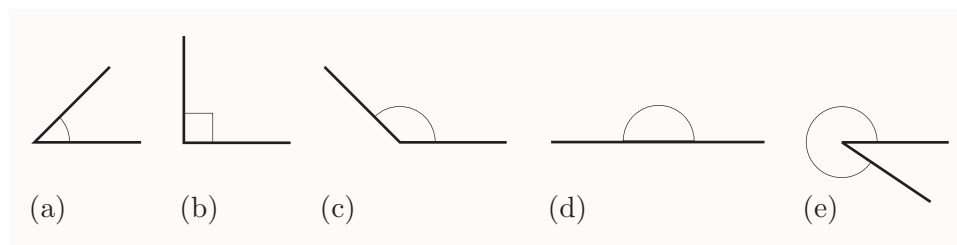


Figure 23 (a) Acute angle (b) right angle (c) obtuse angle (d) straight angle (e) reflex angle

You can also have angles larger than 360° (2π). For example, if a wheel rotates through a full turn followed by another half turn, then altogether it has rotated through an angle of 540° (3π). You can even have negative angles. For example, if a wheel rotates through a half turn followed by another half turn in the opposite direction, then you can say that it has rotated through 180° (π) followed by -180° ($-\pi$).

In this section you'll learn about the sine, cosine and tangent of every angle, even negative angles. The first step is to learn to associate with each angle θ a particular point P on the **unit circle**, which is the circle with radius 1 centred on the origin. The position of the point P is determined by the angle θ : it is obtained by a rotation around the origin through the angle θ , starting from the point on the x -axis with x -coordinate 1. If the angle θ is positive, then the rotation is anticlockwise; if θ is negative, then the rotation is clockwise.

Figure 24 shows some examples of how angles give rise to points on the unit circle.

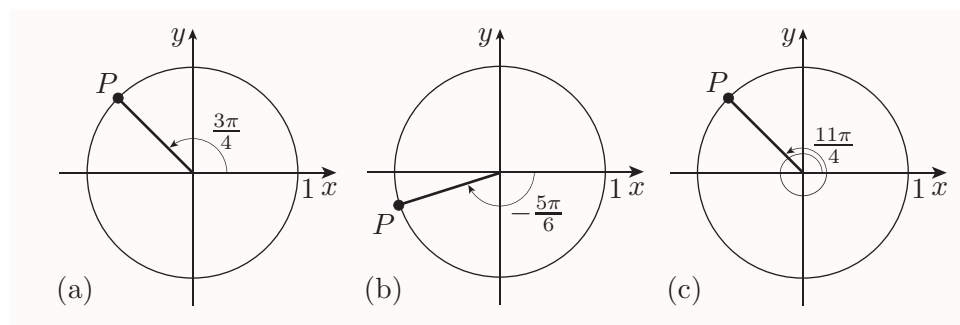


Figure 24 Angles and their associated points on the unit circle

The point P lies in the same position in Figures 24(a) and (c), despite having rotated through different angles, $3\pi/4$ and $11\pi/4$. This is because $11\pi/4$ is exactly one full turn more than $3\pi/4$, since

$$\frac{11\pi}{4} = \frac{3\pi}{4} + 2\pi.$$

In general, whenever two angles differ by an even multiple of π (such as 4π , -6π or 100π), they are both associated with the same point P on the unit circle. For instance, because

$$\frac{3\pi}{4} = -\frac{13\pi}{4} + 4\pi,$$

the point P associated with the angle $-13\pi/4$ is the same as the point P associated with the angle $3\pi/4$.

Skateboard tricks

Skateboarders sometimes perform tricks in which they launch themselves and their board into the air and rotate as much as they can before landing. The name of the trick depends on the amount that they rotate in degrees. For example, a one-full-turn trick is called a 360, a two-full-turn trick is called a 720, and a one-and-a-half-full-turn trick is called a 540. The landing positions of some of these tricks are the same: following a 180, a 540 or a 900, a skateboarder lands in the opposite direction from which he started, because these three angles differ by multiples of 360.



Skateboarders seem unaware of radians!

To help us describe the approximate location of the point P associated with an angle, it is useful to introduce the idea of a *quadrant*. A **quadrant** is one of the four regions separated off by the x -axis and y -axis. The names of the quadrants are shown in Figure 25.

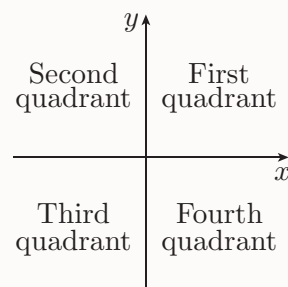


Figure 25 The four quadrants

So, for example, in Figures 24(a) and (c) the point P lies in the second quadrant, whereas in Figure 24(b) it lies in the third quadrant.

Activity 12 *Plotting points on the unit circle associated with angles*

For each of the following angles, draw a diagram similar to those in Figure 24 that shows the location of the point P on the unit circle, and state the quadrant in which P lies.

- (a) $-\pi/3$ (b) $6\pi/5$ (c) $11\pi/3$ (d) $-16\pi/5$

2.2 Sine, cosine and tangent of any angle

In this subsection you'll learn how the sine, cosine and tangent of any angle are defined. Before seeing the definition, consider any *acute* angle θ and its associated point P on the unit circle, as illustrated in Figure 26(a). A line has been drawn from P to the x -axis, to complete a right-angled triangle. A close-up of this triangle is shown in Figure 26(b).

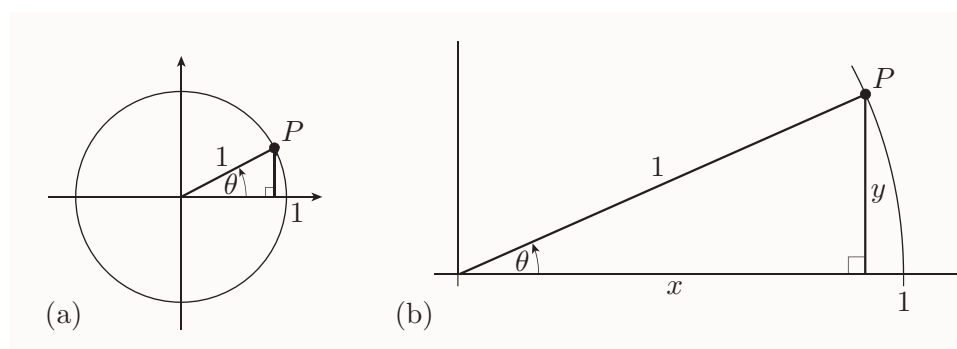


Figure 26 (a) The point P on the unit circle associated with an acute angle θ (b) a close-up of the right-angled triangle

Suppose that the coordinates of the point P are (x, y) . Then you can see from Figure 26 that

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y,$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x,$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}.$$

These equations are used to define the sine, cosine and tangent of *any* angle θ , as follows.

Sine, cosine and tangent

Suppose that θ is any angle and (x, y) are the coordinates of its associated point P on the unit circle. Then

$$\sin \theta = y, \quad \cos \theta = x,$$

and, provided that $x \neq 0$,

$$\tan \theta = \frac{y}{x}.$$

(If $x = 0$, then $\tan \theta$ is undefined.)

In other words, the sine and cosine of an angle θ are just the y - and x -coordinates, in that order, of the point P on the unit circle associated with θ . The tangent of θ is the y -coordinate of P divided by the x -coordinate, provided that the x -coordinate isn't zero. You can see a demonstration of these definitions in the *Sine, cosine and tangent of any angle* applet.

For example, Figure 27(a) shows the point P associated with the angle $\theta = 0$. It has coordinates $(1, 0)$, which tells you that

$$\sin 0 = 0, \quad \cos 0 = 1 \quad \text{and} \quad \tan 0 = \frac{0}{1} = 0.$$

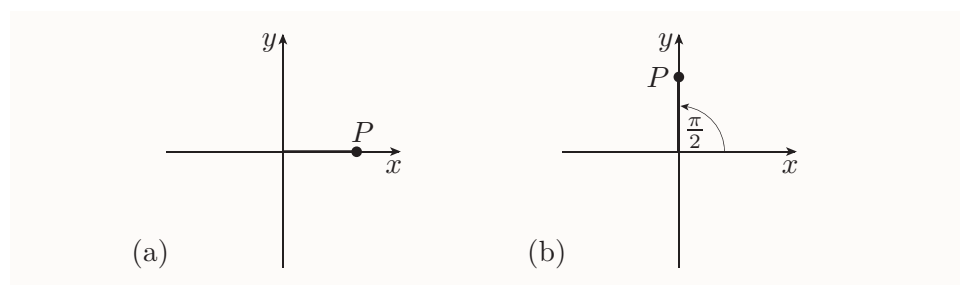


Figure 27 The points on the unit circle associated with the angles (a) 0 (b) $\pi/2$

Similarly, Figure 27(b) shows the point P associated with the angle $\theta = \pi/2$. It has coordinates $(0, 1)$, which tells you that

$$\sin \frac{\pi}{2} = 1, \quad \cos \frac{\pi}{2} = 0 \quad \text{and} \quad \tan \frac{\pi}{2} \text{ is undefined.}$$

Activity 13 Finding sines, cosines and tangents

Find the following values, where they are defined, without using a calculator.

- (a) $\sin \pi$, $\cos \pi$ and $\tan \pi$
 (b) $\sin(-\pi/2)$, $\cos(-\pi/2)$ and $\tan(-\pi/2)$

You can see that it's straightforward to write down the sine, cosine and tangent (if it's defined) of any angle whose associated point P lies on one of the coordinate axes – that is, any angle that's an integer multiple of $\pi/2$.

Notice also that if the point P associated with an angle θ lies on the y -axis, then the tangent of θ is undefined (because at such points the value of x is zero). So every angle of the form

$$\frac{\pi}{2} + n\pi, \quad \text{where } n \text{ is an integer,}$$

has no tangent.

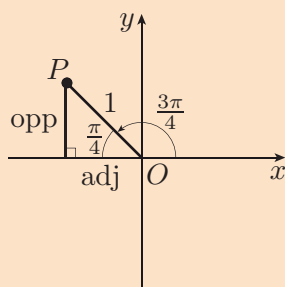
The next example shows you a method for finding the sine, cosine and tangent of any angle θ whose associated point P doesn't lie on one of the coordinate axes. The method is based on first finding the sine, cosine and tangent of a particular *acute* angle that's related to θ .

Example 4 Finding the sine, cosine and tangent of an angle that's not an integer multiple of $\pi/2$

Find the sine, cosine and tangent of $3\pi/4$.

Solution

 Draw a sketch showing the approximate position of the associated point P (the important thing is to draw it in the correct quadrant). Work out and mark the size of the acute angle between the line OP and the x -axis. Draw a line from P to the x -axis, at right angles to the x -axis. 



Use basic trigonometry to find the lengths of the adjacent and opposite sides of the resulting right-angled triangle. Hence find the coordinates of P , and write down the sine, cosine and tangent of the original angle.

For the triangle in the diagram,

$$\cos \frac{\pi}{4} = \frac{\text{adj}}{1} = \text{adj}, \quad \text{so} \quad \text{adj} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\sin \frac{\pi}{4} = \frac{\text{opp}}{1} = \text{opp}, \quad \text{so} \quad \text{opp} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hence the coordinates of P are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Therefore

$$\sin \left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}},$$

$$\cos \left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}},$$

$$\tan \left(\frac{3\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right) / \left(-\frac{1}{\sqrt{2}}\right) = -1.$$

In the next activity, you can practise using the method demonstrated in Example 4. When you use this method, you should always find the size of the acute angle between OP and the x -axis (never the y -axis).

Activity 14 Finding the sine, cosine and tangent of an angle that's not an integer multiple of $\pi/2$

Find the sine, cosine and tangent of the following angles.

- (a) $5\pi/6$ (b) $-\pi/4$

You can use the special angles table on page 22 (it is also in the *Handbook*) to find the sines, cosines and tangents of acute angles that you'll need.

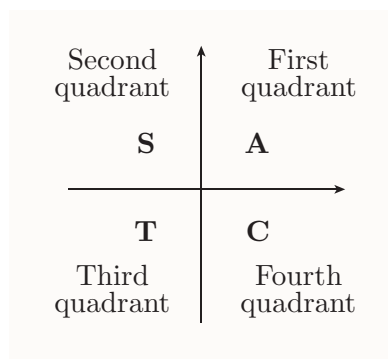


Figure 28 The ASTC diagram

There's a convenient way to shorten the working when you use the method demonstrated in Example 4. It's based on the diagram in Figure 28. In this diagram the letter in each quadrant indicates which of $\sin \theta$, $\cos \theta$ and $\tan \theta$ are positive when the point P associated with the angle θ lies in that quadrant:

- A stands for all
- S stands for sin
- T stands for tan
- C stands for cos.

This diagram, which we'll call the *ASTC diagram*, is often useful, so it's a good idea to memorise it, if you can. You might find it helpful to use a mnemonic, such as 'All Silly Tom Cats'. In some texts it's called the CAST, rather than ASTC, diagram, which is easier to remember, but has the disadvantage that the first letter of the acronym doesn't correspond to the first quadrant. To use the ASTC diagram, you combine it with the following useful fact, which is illustrated by the working in Example 4 and Activity 14. (Here the symbol ϕ is the lower-case form of the Greek letter phi, which is pronounced 'fie'.)

Suppose that θ is an angle whose associated point P does not lie on either the x - or y -axis, and ϕ is the acute angle between OP and the x -axis. Then

$$\begin{aligned}\sin \theta &= \pm \sin \phi \\ \cos \theta &= \pm \cos \phi \\ \tan \theta &= \pm \tan \phi.\end{aligned}$$

The ASTC diagram tells you which sign applies in each case.

(The values $\sin \phi$, $\cos \phi$ and $\tan \phi$ are all positive, because ϕ is acute.)

The next example repeats Example 4 above, but uses the ASTC diagram to shorten the working.

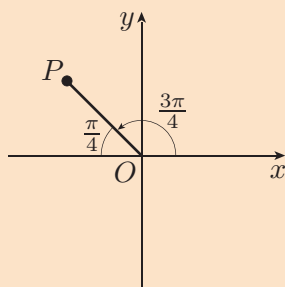


Example 5 Finding values of sine, cosine and tangent using the ASTC diagram

Find the sine, cosine and tangent of $3\pi/4$.

Solution

Draw a sketch showing the approximate position of the associated point P (the important thing is to draw it in the correct quadrant). Use it to work out the size of the acute angle between OP and the x -axis.



Use the ASTC diagram to express the sine, cosine and tangent of the original angle in terms of the sine, cosine and tangent of the acute angle. Then obtain the final answers by finding the sine, cosine and tangent of the acute angle. Here you can use the special angles table.

Since $3\pi/4$ is a second quadrant angle, its sine is positive and its cosine and tangent are negative. Hence

$$\sin\left(\frac{3\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}},$$

$$\tan\left(\frac{3\pi}{4}\right) = -\tan\frac{\pi}{4} = -1.$$

Activity 15 Finding trigonometric ratios using the ASTC diagram

Use the ASTC diagram and the special angles table (page 22) to find the sine, cosine and tangent of each of the following angles.

- (a) $2\pi/3$ (b) $7\pi/6$ (c) $-7\pi/3$

Another way to find the sine, cosine or tangent of any angle is simply to use your calculator, in the same way that you do for acute angles. You might like to use your calculator to check your answers to Activity 15.

However, it's still important that you understand the method demonstrated in Examples 4 and 5. One reason for this is that you can sometimes use this method to obtain exact values of sine, cosine and tangent where a calculator will give you only approximate values. Another reason is that you can use the ideas of the method to solve trigonometric equations, as you'll see later in this section.

2.3 Graphs of sine, cosine and tangent

You have seen that sine, cosine and tangent are rules whose inputs are the sizes of angles and whose outputs are numbers. In other words, sine, cosine and tangent are *functions*. They are known as **trigonometric functions**.

You may think at first that the trigonometric functions aren't quite the same as the functions that you met in Unit 3. For a start, functions in Unit 3 were usually written as a single letter, such as f , whereas the sine, cosine and tangent functions are written as three letters, namely $\sin$, $\cos$ and $\tan$.

Another difference is that you wrote $f(x)$ in Unit 3 for the image of an element under a function, whereas here we have been writing $\sin \theta$, $\cos \theta$ and $\tan \theta$. This is a convention in trigonometry: angles are usually represented by θ in preference to x , and brackets are usually omitted from expressions such as $\sin \theta$. It is also perfectly correct to write $\sin(\theta)$, or to represent an angle by x and write $\sin x$ or $\sin(x)$.

Another apparent difference is that the input values of the trigonometric functions are angles, rather than real numbers. In fact, you can think of the input values of trigonometric functions as real numbers, and it's often convenient to do so. The notation that we've been using for angle sizes helps us to do this. For example, when we write $\sin(\pi/2)$, the input value $\pi/2$ is a real number that is the size of an angle measured in radians. In general, the input values of the trigonometric functions are real numbers that are the sizes of angles measured in radians. (However, it still makes sense to write $\sin(90^\circ)$, for example, because $90^\circ = \pi/2$.)

As you saw in Unit 3, a useful way to understand the properties of a function is to draw its graph. In this subsection you'll see the graphs of the trigonometric functions, and use these graphs as an alternative to the ASTC diagram to work out values of sine, cosine and tangent.

In the next activity, you can see the graphs of the sine and cosine functions generated directly from their definitions in terms of the coordinates of a point on the unit circle.



Activity 16 Understanding graphs of trigonometric functions

Open the *Graphs of sine and cosine* applet. Explore the graphs of the sine and cosine functions as directed.

Let's now look at some of the properties of the graphs of the sine, cosine and tangent functions in more detail.

The graph of the sine function

The graph of sine is shown in Figure 29.

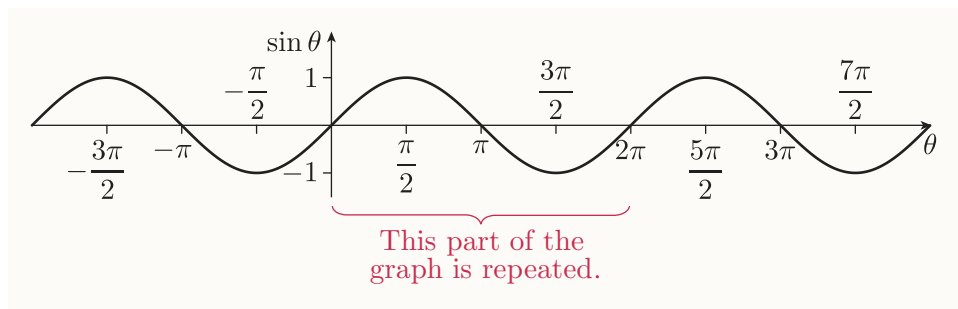


Figure 29 The graph of the sine function

First look at the part of the graph for values of θ between 0 and 2π . As the angle θ increases from 0 to 2π , the value of $\sin \theta$ oscillates from 0 to 1 to 0 to -1 and back to 0. To see why this is, remember that $\sin \theta$ is the y -coordinate of the point P on the unit circle (see Figure 30). As the angle θ increases from 0 to 2π , the point P starts on the positive x -axis and rotates once anticlockwise around the unit circle, back to where it started. As P rotates, its y -coordinate oscillates from 0 to 1 to 0 to -1 and back to 0. Nowhere is $\sin \theta$ greater than 1 or less than -1 , because P is constrained to lie on the unit circle.

Since P returns to its original position after a complete rotation through 2π , you can see that $\sin(\theta + 2\pi) = \sin \theta$ for any angle θ . This implies that the graph of sine oscillates endlessly to the right and to the left, repeating its shape after every interval of 2π . This fact is expressed by saying that the graph is **periodic**, with **period** 2π .

The graph of sine has other symmetry features. For instance, if you rotate the whole graph through a half-turn about the origin, then it remains unchanged. Also, any vertical line through the centre of a peak or trough of the graph is a line of mirror symmetry.

The graph of the cosine function

The graph of cosine is shown in Figure 31.

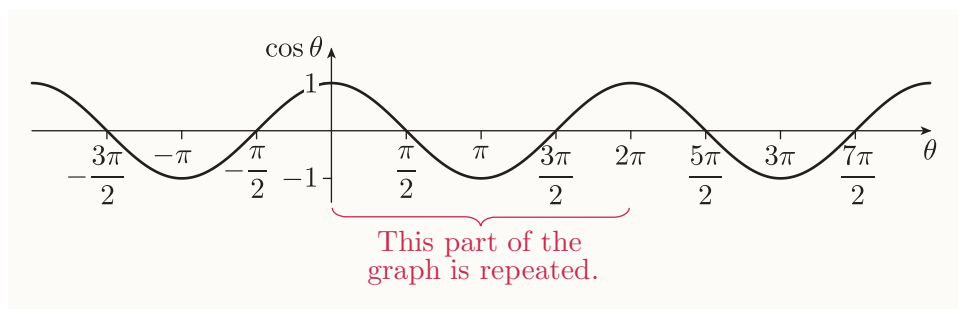


Figure 31 The graph of the cosine function

To understand why the graph of cosine has the shape that it does, you can think about it in a similar way to the graph of sine, remembering that

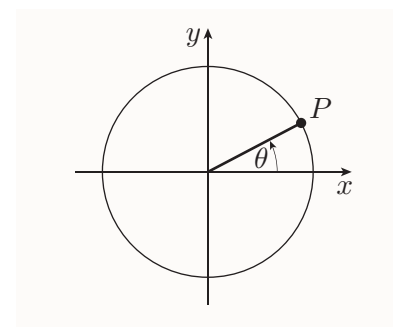


Figure 30 The point P has coordinates $(\cos \theta, \sin \theta)$

$\cos \theta$ is the x -coordinate of the point P on the unit circle, rather than the y -coordinate. Like the graph of sine, the graph of cosine is periodic with period 2π , and any vertical line through the centre of a peak or trough is a line of mirror symmetry. In fact, you can obtain the graph of cosine by translating the graph of sine to the left by $\pi/2$.

The graph of the tangent function

The graph of the tangent function, shown in Figure 32, differs from the other two graphs in that it is broken up into separate pieces, and it takes arbitrarily large positive values and arbitrarily large negative values.

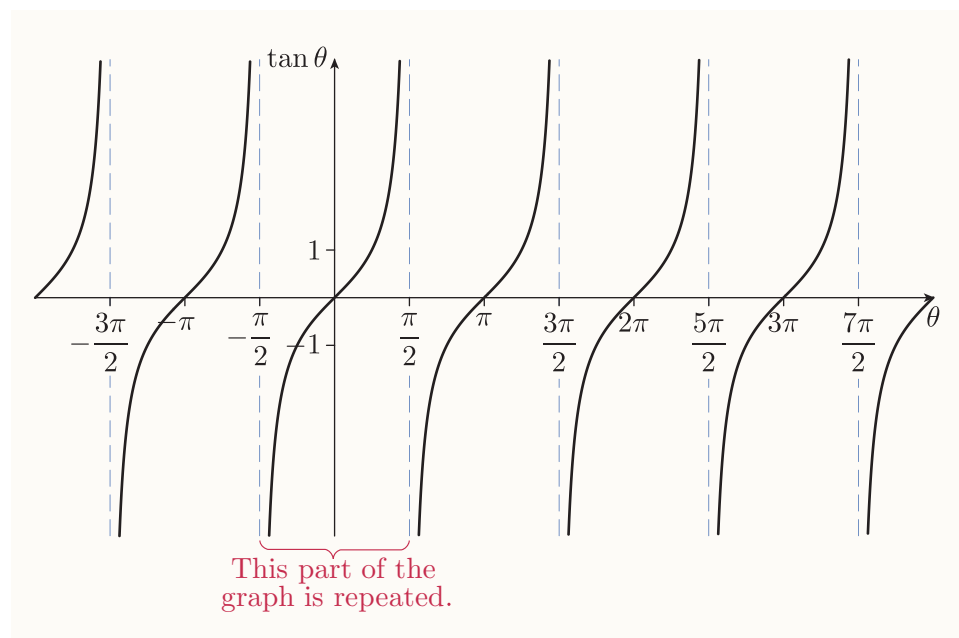


Figure 32 The graph of the tangent function

Like the graphs of sine and cosine, the graph of tangent is periodic, but its period is π rather than 2π . The breaks in the graph occur when $\theta = \pi/2 + n\pi$, for some integer n . These are the values of θ for which $\tan \theta$ is undefined, because the y -coordinate of the point P is zero, as you saw on page 30. The vertical dashed lines drawn on the graph are asymptotes: the graph approaches but never reaches them. (You met the idea of an asymptote in Unit 3.)

Using the graphs to find values of trigonometric functions

You can use the graphs that you've met in this subsection to work out the sine, cosine or tangent of an angle from the sine, cosine or tangent of an acute angle, as an alternative to using the ASTC diagram. The method is demonstrated in the following example.

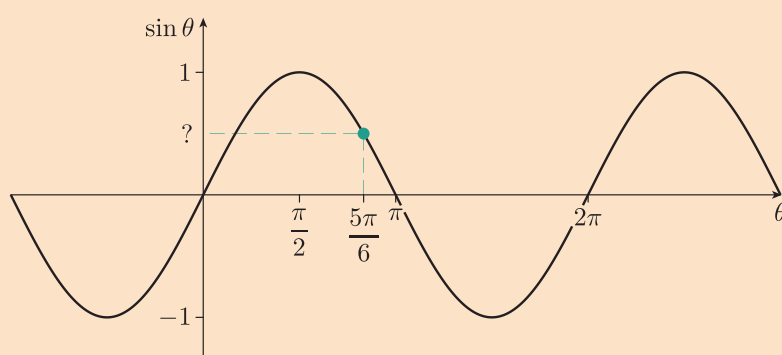
Example 6 Finding sines, cosines and tangents by using graphs

Use the graphs of sine and tangent, and the special angles table, to find the following values.

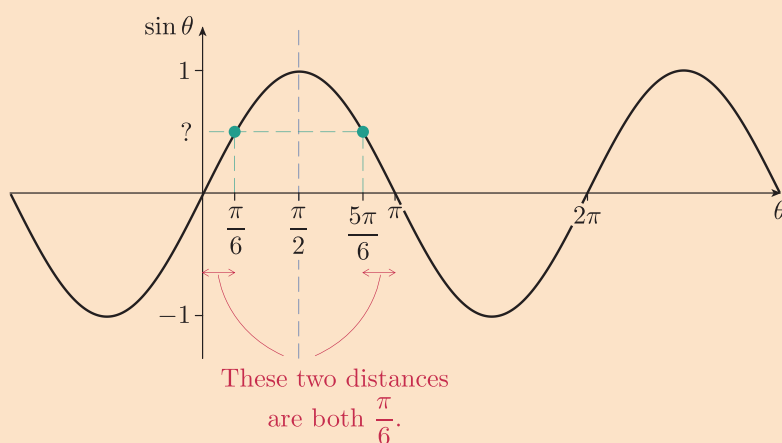
(a) $\sin\left(\frac{5\pi}{6}\right)$ (b) $\tan\left(-\frac{\pi}{3}\right)$

Solution

- (a) Find the approximate location of $5\pi/6$ on the θ -axis of the graph of sine (the important thing is to locate it between $\pi/2$ and π). Mark the corresponding point on the graph.



- Use the symmetry of the graph to relate $\sin(5\pi/6)$ to the sine of an acute angle. You can use the vertical line of symmetry through $\theta = \pi/2$.



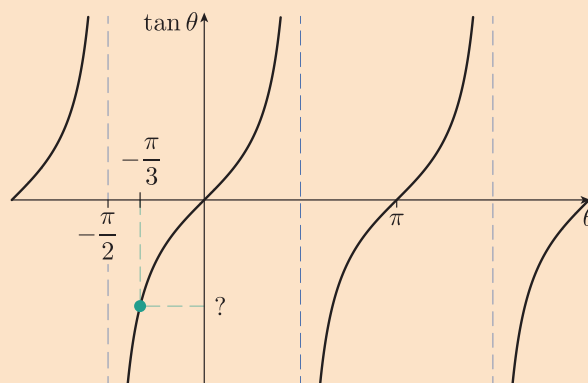
From the graph,

$$\sin\left(\frac{5\pi}{6}\right) = \sin\frac{\pi}{6}.$$

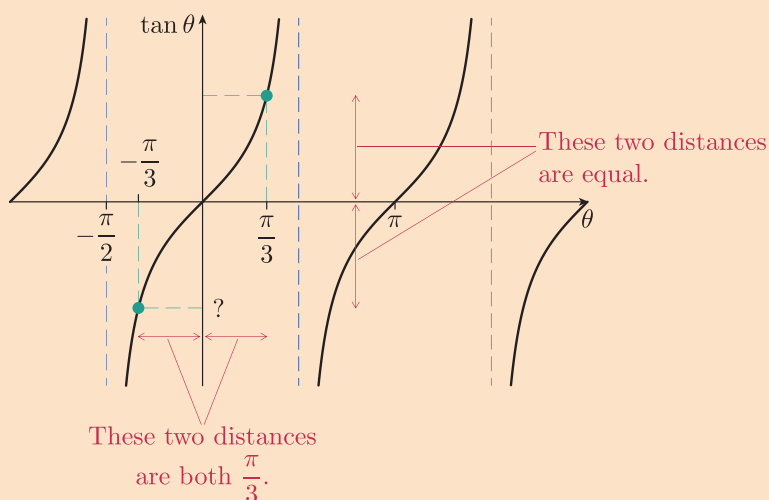
Since $\sin(\pi/6) = \frac{1}{2}$, it follows that

$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}.$$

- (b) Find the approximate location of $-\pi/3$ on the θ -axis of the graph of tangent (the important thing is to locate it between $-\pi/2$ and 0). Mark the corresponding point on the graph.



- Use the symmetry of the graph to relate $\tan(-\pi/3)$ to the tangent of an acute angle.



From the graph,

$$\tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3}.$$

Since $\tan(\pi/3) = \sqrt{3}$, it follows that

$$\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$$

In the next activity, you can practise finding values of sine, cosine and tangent using the graph method demonstrated in Example 6. The graphs that you sketch don't need to be accurate: they just need to illustrate the symmetry properties of the trigonometric functions.

Activity 17 Finding sines, cosines and tangents by using graphs

Use the graphs of sine, cosine and tangent to find the following values.

(a) $\sin\left(-\frac{\pi}{4}\right)$ (b) $\cos\left(-\frac{\pi}{3}\right)$ (c) $\tan\left(\frac{5\pi}{4}\right)$

Using the variables x and y for trigonometric functions

Sometimes it's convenient to use a letter other than θ to denote the input variable of a trigonometric function. For example, it's often convenient to use the letter x to denote the input variable and the letter y to denote the output variable. These are the standard letters for the input and output variables of functions, as you saw in Unit 3. You'll use this notation in the next subsection, and also in the calculus units later in the module.

Figure 33 shows the graphs of the trigonometric functions with this notation used.

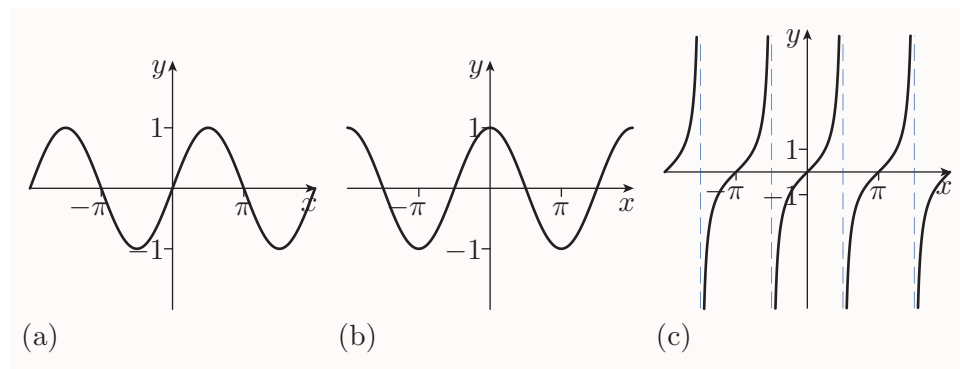


Figure 33 The graphs of (a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

You need to be careful with this change of notation. Earlier in this unit, the letters x and y denoted the coordinates of the point P on the unit

circle. With the changed notation, we're using the same letters for completely different quantities. In particular, remember that x denotes the quantity that was previously denoted by θ .



An electrocardiogram displays the heart's electrical activity, which varies periodically with every heartbeat

Modelling waves

In physics, a *wave* is an oscillation that transfers energy from one place to another. There are waves that you can see, such as water waves and waves on a guitar string, and waves that are invisible to the human eye, such as sound waves and microwaves. When waves are invisible, imaging devices can be used to represent them graphically.

Simple periodic waves can be modelled by translating and scaling the sine or cosine function. However, periodic waves may have a more complicated form, as illustrated in Figure 34.

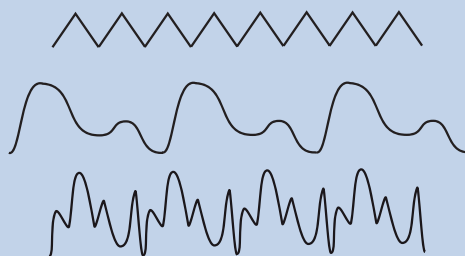


Figure 34 A selection of waves

You can model complicated periodic waves by scaling sine and cosine functions and adding them together. For example, Figure 35 shows the graph of the function

$$f(t) = 1.4 \sin(t) + 0.3 \cos(2t) + \sin(5t).$$

In this case, the letter t is used for the input variable of the function, rather than θ . This is because a function that models a wave often has an input variable that represents time.

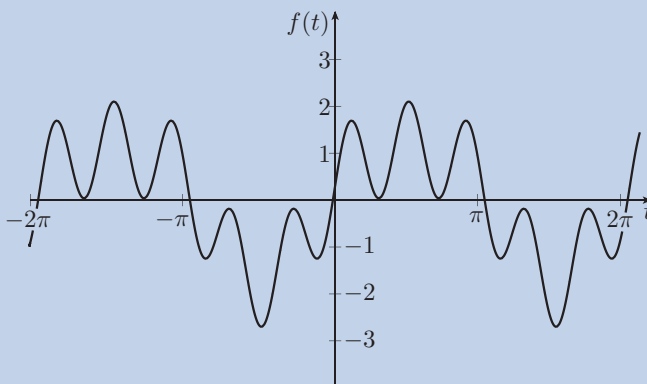


Figure 35 The graph of $f(t) = 1.4 \sin(t) + 0.3 \cos(2t) + \sin(5t)$

If you add together many scaled sine and cosine functions, then you obtain a type of function called a *Fourier series*. Such functions can be used to model *any* periodic wave. They were developed by the French mathematician Joseph Fourier (1768–1830), in order to model variations in temperatures.

2.4 Inverse trigonometric functions

In this subsection you'll learn more about the *inverse trigonometric functions* $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, which you used in Subsection 1.4 to find acute angles in right-angled triangles.

In Unit 3 you saw that if a function f is one-to-one, then it has an *inverse function* f^{-1} , whose rule is given by

$$f^{-1}(y) = x, \quad \text{where} \quad f(x) = y.$$

The sine, cosine and tangent functions are not one-to-one. Take the sine function, for example, whose graph is shown in Figure 36. You can see that there are many input numbers that have the output number $\frac{1}{2}$, such as $\pi/6$, $5\pi/6$ and $-7\pi/6$. In fact, because the sine function is periodic, there are infinitely many input numbers with the output number $\frac{1}{2}$.

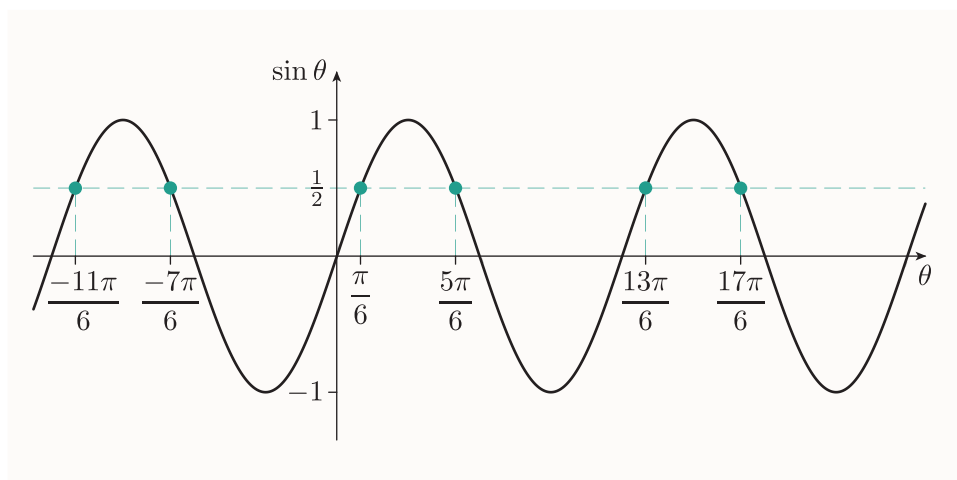


Figure 36 The graph of the sine function, with solutions of $\sin x = \frac{1}{2}$ labelled

As you saw in Unit 3, to obtain ‘some sort of inverse function’ for a function that isn’t one-to-one, we need to start by specifying a new function that has the same rule as the original function, but a smaller domain. (Remember that such a function is called a *restriction* of the original function.) We choose the smaller domain to ensure that the new function is one-to-one and has the same image set as the original function.

For the sine function, the most convenient smaller domain is the interval $[-\pi/2, \pi/2]$, which includes all the acute angles. The graph of the sine function with this restricted domain is shown in Figure 37.

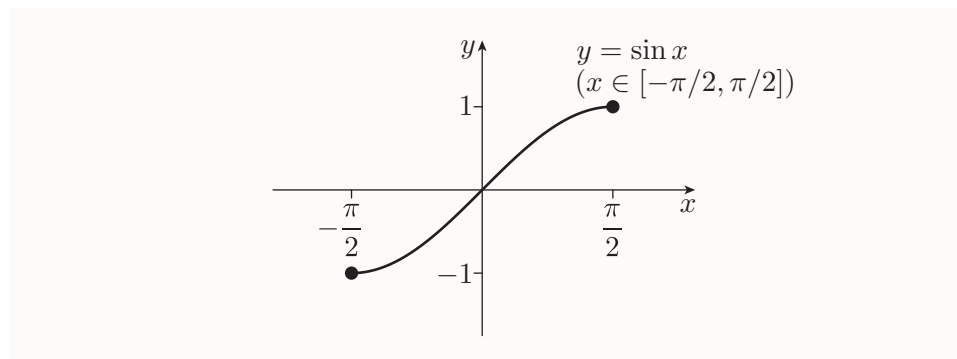


Figure 37 The graph of $y = \sin x$ ($x \in [-\pi/2, \pi/2]$)

The function in Figure 37 has the same image set as the sine function, namely the interval $[-1, 1]$. It is increasing on the whole of its domain, so it is one-to-one, and therefore has an inverse function. Its inverse function is called the **inverse sine function** and is denoted by $\sin^{-1}$. (It is convenient to use this terminology and notation, even though $\sin^{-1}$ is actually the inverse function of a restricted version of the sine function.)

Inverse sine function

The **inverse sine function** $\sin^{-1}$ is the function with domain $[-1, 1]$ and rule

$$\sin^{-1} x = y,$$

where y is the number in the interval $[-\pi/2, \pi/2]$ such that $\sin y = x$.

As before, you may choose to write either $\sin^{-1} x$ or $\sin^{-1}(x)$, whichever seems clearer.

You can find the inverse sines of some numbers by using the sines of angles that you know already. For example,

$$\sin 0 = 0, \quad \text{so} \quad \sin^{-1} 0 = 0,$$

$$\sin \frac{\pi}{2} = 1, \quad \text{so} \quad \sin^{-1} 1 = \frac{\pi}{2},$$

and

$$\sin \frac{\pi}{6} = \frac{1}{2}, \quad \text{so} \quad \sin^{-1} \frac{1}{2} = \frac{\pi}{6}.$$

However, you have to be careful. For example, $\sin(5\pi/6) = \frac{1}{2}$, but it doesn't follow that $\sin^{-1} \frac{1}{2} = 5\pi/6$, because $5\pi/6$ isn't in the interval $[-\pi/2, \pi/2]$. In fact, $\sin^{-1} \frac{1}{2} = \pi/6$, as stated above. There's more about finding the inverse sines of numbers (and inverse cosines and inverse tangents) later in this subsection, and in Subsection 2.5.

You saw in Unit 3 that the graphs of a one-to-one function and its inverse function are the reflections of each other in the line $y = x$ (when the coordinate axes have equal scales). So the graph of the inverse sine function is as shown in Figure 38(b).

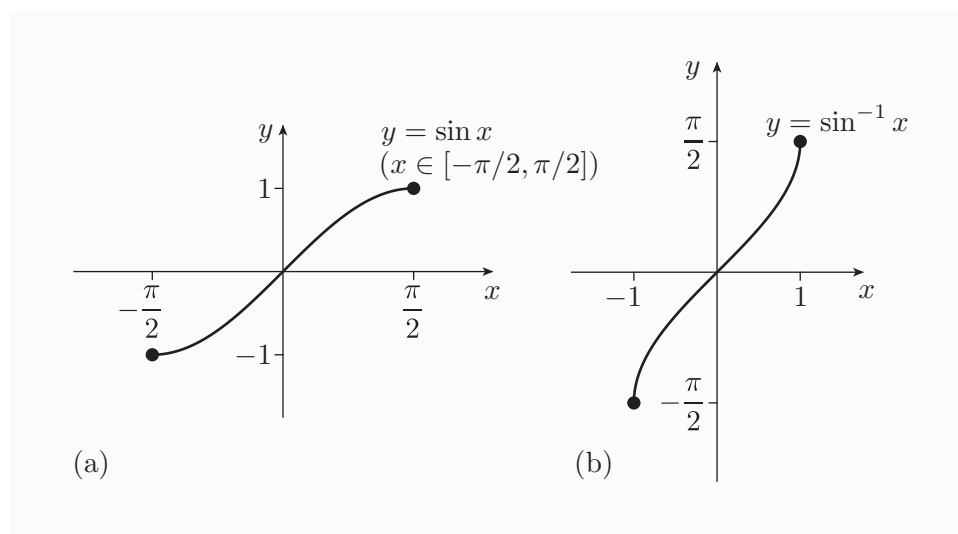


Figure 38 The graphs of (a) $y = \sin x$ ($x \in [-\pi/2, \pi/2]$) (b) $y = \sin^{-1} x$

Like the sine function, the cosine function is not one-to-one, so to obtain an inverse function we must restrict its domain. The most convenient smaller domain for the cosine function is the interval $[0, \pi]$, which again includes all the acute angles. The graph of the function obtained by restricting the domain of the cosine function to this interval is shown in Figure 39(a). It is decreasing on the whole of its domain, so it has an inverse function, called the **inverse cosine function** and denoted by $\cos^{-1}$.

Inverse cosine

The **inverse cosine function** $\cos^{-1}$ is the function with domain $[-1, 1]$ and rule

$$\cos^{-1} x = y,$$

where y is the number in the interval $[0, \pi]$ such that $\cos y = x$.

The graph of the inverse cosine function is obtained by reflecting the graph in Figure 39(a) in the line $y = x$, and is shown in Figure 39(b).

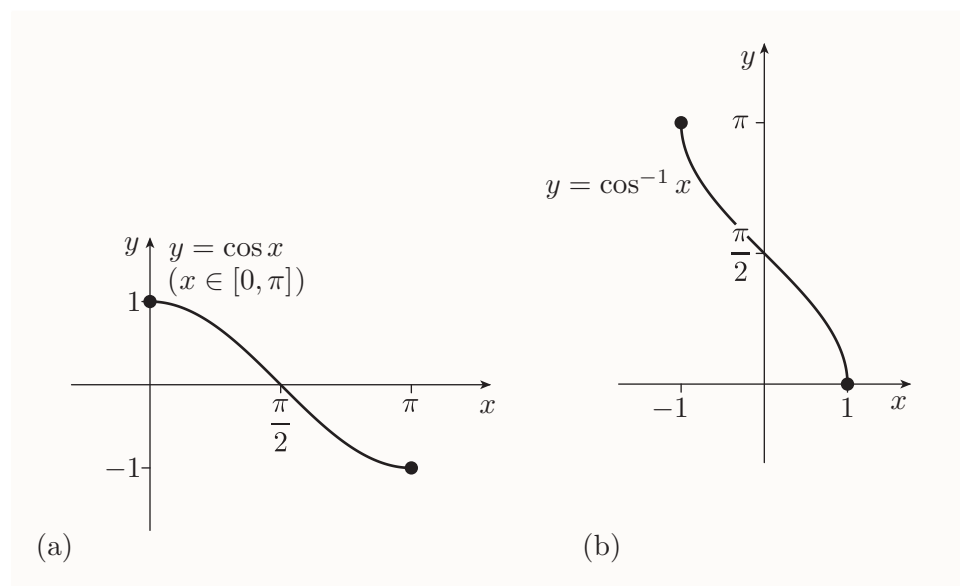


Figure 39 The graphs of (a) $y = \cos x$ ($x \in [0, \pi]$) (b) $y = \cos^{-1} x$

We can carry out a similar process for the tangent function. The most convenient smaller domain for this function is the interval $(-\pi/2, \pi/2)$. The function obtained by restricting the tangent function to this domain is shown in Figure 40(a). Its inverse function is called the **inverse tangent function**, denoted by $\tan^{-1}$, and its graph is shown in Figure 40(b).

Inverse tangent

The **inverse tangent function** $\tan^{-1}$ is the function with domain $\mathbb{R}$ and rule

$$\tan^{-1} x = y,$$

where y is the number in the interval $(-\pi/2, \pi/2)$ such that $\tan y = x$.

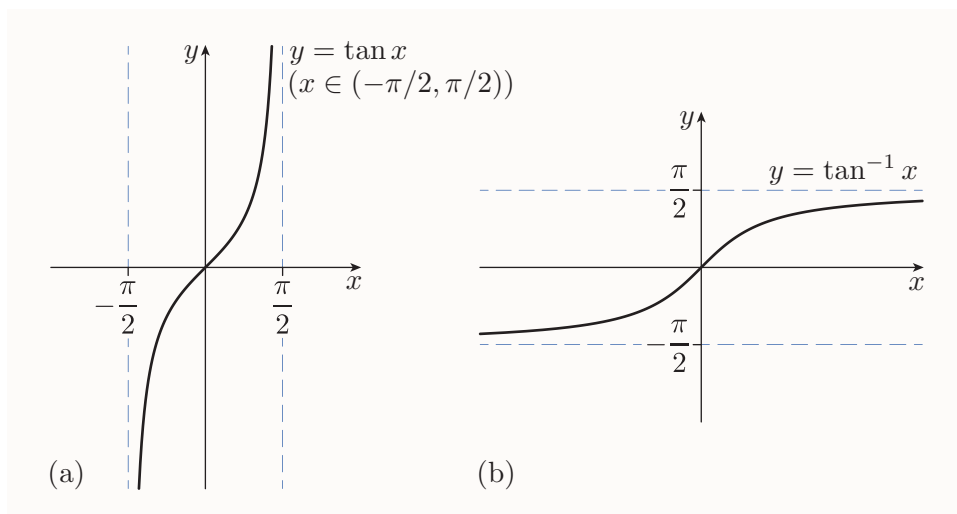


Figure 40 The graphs of (a) $y = \tan x$ ($x \in (-\pi/2, \pi/2)$) (b) $y = \tan^{-1} x$

The graph in Figure 40(a) has two vertical asymptotes, with equations $x = -\pi/2$ and $x = \pi/2$. Accordingly, the graph of the inverse tangent function, in Figure 40(b), has two horizontal asymptotes, with equations $y = -\pi/2$ and $y = \pi/2$.

The inverse sine, inverse cosine and inverse tangent functions are known as **inverse trigonometric functions**. As you saw earlier, alternative names for these three functions are arcsin, arccos and arctan, respectively. Calculators and computers may also use the names INV SIN, INV COS, INV TAN or asin, acos, atan.

One way to find the inverse sine, inverse cosine or inverse tangent of a number is to use your calculator. You did this for positive numbers in Subsection 1.4, and you can find the inverse sine, inverse cosine or inverse tangent of any suitable number in the same way. For example, a calculator gives $\cos^{-1}(-0.6) = 2.214\,297\,43\dots$

However, a calculator may give a decimal approximation where it is possible to find an exact value. As you saw earlier, you can find the exact values of the inverse sine, inverse cosine or inverse tangent of some special numbers by using values of sine, cosine and tangent that you know already.

For example, you know from the special angles table on page 22 that $\tan(\pi/6) = 1/\sqrt{3}$. So, since the angle $\pi/6$ lies in the interval to which we restricted the domain of the tangent function earlier, namely $(-\pi/2, \pi/2)$, it follows that $\tan^{-1}(1/\sqrt{3}) = \pi/6$.

Activity 18 *Finding exact values of inverse sines, inverse cosines and inverse tangents*

Find the following angles without using your calculator.

- (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1} \frac{1}{2}$ (c) $\cos^{-1} 0$ (d) $\cos^{-1} 1$
 (e) $\cos^{-1}(\sqrt{3}/2)$ (f) $\tan^{-1} 0$ (g) $\tan^{-1} 1$

All the values of sine, cosine and tangent in the special angles table are positive, so you can't use this table directly to find the exact values of the inverse sine, inverse cosine or inverse tangent of any negative numbers. However, you can find these values by using symmetry properties of the inverse trigonometric functions that follow from the symmetry properties of the trigonometric functions.

For example, as you can see from the graph of $y = \sin^{-1} x$ in Figure 38 on page 43, the following identity holds:

$$\sin^{-1}(-x) = -\sin^{-1} x.$$

So, for instance, since $\sin^{-1} \frac{1}{2} = \pi/6$, you can deduce that

$$\sin^{-1} \left(-\frac{1}{2} \right) = -\sin^{-1} \frac{1}{2} = -\frac{\pi}{6}.$$

The graphs of $y = \cos^{-1} x$ and $y = \tan^{-1} x$, in Figures 39 and 40, also have symmetry properties that give rise to similar identities. These identities are listed below.

$$\begin{aligned}\sin^{-1}(-x) &= -\sin^{-1} x \\ \cos^{-1}(-x) &= \pi - \cos^{-1} x \\ \tan^{-1}(-x) &= -\tan^{-1} x\end{aligned}$$

These identities enable you to find inverse sines, inverse cosines and inverse tangents of negative numbers. The following activity gives you a chance to try this.

Activity 19 *Finding further exact values of inverse sines, inverse cosines and inverse tangents*

Use the identities above and your answers to Activity 18 to find the following angles without using your calculator.

- (a) $\sin^{-1}(-\sqrt{3}/2)$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\cos^{-1}(-1)$
 (d) $\cos^{-1}(-\sqrt{3}/2)$ (e) $\tan^{-1}(-1)$

2.5 Solving simple trigonometric equations

Equations that contain trigonometric functions of the unknown(s) are called **trigonometric equations**. Simple trigonometric equations such as

$$\sin \theta = \frac{1}{2}, \quad \cos \theta = 0.3 \quad \text{and} \quad \tan \theta = -1$$

occur frequently when you're working with trigonometric functions.

One solution of such a simple trigonometric equation can be obtained by using an inverse trigonometric function described in the previous subsection. For instance, one solution of the equation $\sin \theta = \frac{1}{2}$ is

$$\theta = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}.$$

As you saw earlier, however, there are other solutions, such as $5\pi/6$ and $-7\pi/6$, as illustrated in Figure 41. In fact there are infinitely many solutions of the equation $\sin \theta = \frac{1}{2}$, since the sine function is periodic.

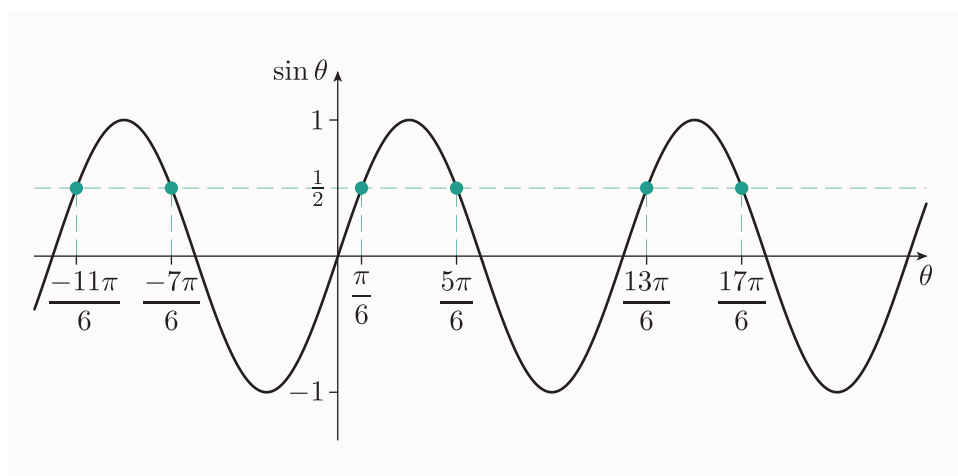


Figure 41 The graph of the sine function, with solutions of $\sin x = \frac{1}{2}$ labelled

Usually when you want to solve a simple trigonometric equation, it's best to begin by finding all the solutions that lie in an interval of length 2π (360°), such as $[0, 2\pi]$ or $[-\pi, \pi]$. There are usually two such solutions. Then you can obtain any other solutions that you want by adding integer multiples of 2π to the first solutions that you found.

In this subsection you'll be shown two methods for finding all the solutions of a simple trigonometric equation that lie in a particular interval of length 2π .

In the first method, you use the ASTC diagram, in Figure 42, and the facts in the box below, which are repeated from page 32.

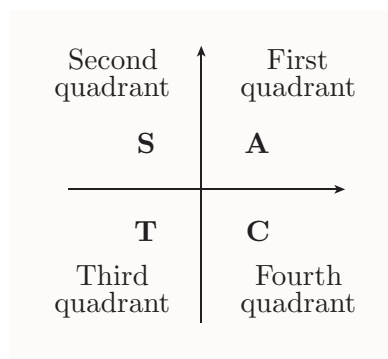


Figure 42 The ASTC diagram

Suppose that θ is any angle whose associated point P does not lie on either the x - or y -axis, and ϕ is the acute angle between OP and the x -axis. Then

$$\sin \theta = \pm \sin \phi$$

$$\cos \theta = \pm \cos \phi$$

$$\tan \theta = \pm \tan \phi.$$

The ASTC diagram tells you which sign applies in each case.

(The values $\sin \phi$, $\cos \phi$ and $\tan \phi$ are all positive, because ϕ is acute.)

The method is illustrated in the next example.



Example 7 Solving simple trigonometric equations using the ASTC diagram

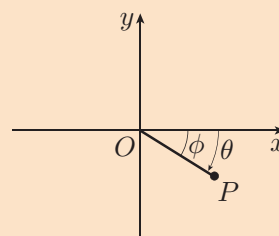
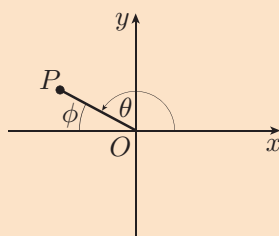
- Find all solutions between $-\pi$ and π of the equation $\tan \theta = -1$. Give exact answers.
- Find all solutions between 0° and 360° of the equation $\cos \theta = 0.3$. Give your answers to the nearest degree.

Solution

- Use the ASTC diagram to find the quadrants of the solutions.

The tangent of θ is negative, so θ is a second- or fourth-quadrant angle.

Sketch each of the possible quadrants with a line OP in each. On each sketch mark the associated angle θ that lies in the interval $[-\pi, \pi]$, and the acute angle ϕ between OP and the x -axis.



Use the given equation and the facts in the box above to write down the value of $\tan \phi$. Hence use the special angles table or your calculator to find ϕ .

Here

$$\tan \theta = -1,$$

so

$$\tan \phi = 1,$$

and hence

$$\phi = \tan^{-1} 1 = \frac{\pi}{4}.$$

Now use your sketches to find the possible values of θ .

The solutions are

$$\theta = \pi - \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

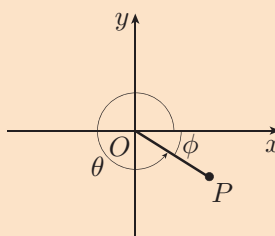
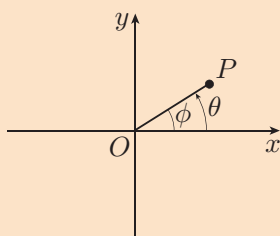
and

$$\theta = -\phi = -\frac{\pi}{4}.$$

- (b) Use the ASTC diagram to find the quadrants of the solutions.

The cosine of θ is positive, so θ is a first- or fourth-quadrant angle.

Sketch each of the possible quadrants with a line OP in each. On each sketch mark the associated angle θ that lies in the interval $[0^\circ, 360^\circ]$, and the acute angle ϕ between OP and the x -axis.



Use the given equation and the facts in the box above to write down the value of $\cos \phi$. Hence use your calculator to find ϕ .

Here

$$\cos \theta = 0.3,$$

so

$$\cos \phi = 0.3,$$

and hence

$$\phi = \cos^{-1} 0.3 = 72.542 \dots^\circ.$$

 Use your sketches to find the possible values of θ . 

The solutions are

$$\theta = \phi = 72.542 \dots^\circ$$

and

$$\begin{aligned}\theta &= 360^\circ - \phi \\ &= 360^\circ - 72.542 \dots^\circ \\ &= 287.457 \dots^\circ.\end{aligned}$$

So the solutions are 73° and 287° , to the nearest degree.

When you use the method demonstrated in Example 7, it is helpful to check your answers. For example, in Example 7(b), you can use your calculator to check that $\cos 73^\circ$ and $\cos 287^\circ$ are both approximately 0.3.

In the next activity, remember to set your calculator to degrees or radians, as appropriate, and to check your answers.

Activity 20 *Solving simple trigonometric equations using the ASTC diagram*

- Find all solutions between $-\pi$ and π of the equation $\sin \theta = -\frac{1}{2}$. Give exact answers. Hence find all solutions between π and 3π of this equation.
- Find all solutions between -180° and 180° of the equation $\tan \theta = -5$. Give your answers to the nearest degree.

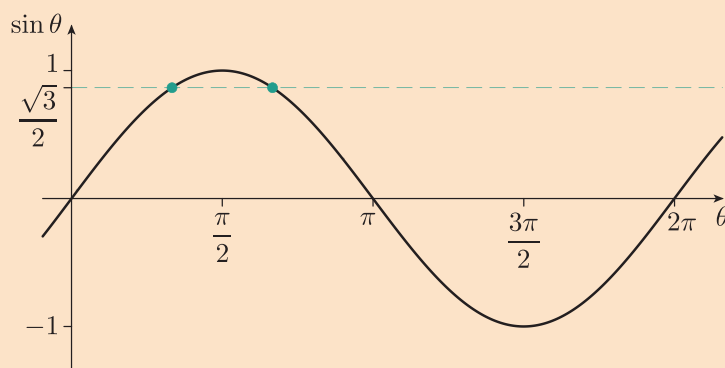
The next example demonstrates an alternative strategy for solving simple trigonometric equations. It involves using the graphs of the trigonometric functions. Usually it doesn't matter which of the two methods you use to solve a simple trigonometric equation: you should choose the one that you prefer.

Example 8 Solving simple trigonometric equations using the graphs of the trigonometric functions

- (a) Find all solutions between 0 and 2π of the equation $\sin \theta = \sqrt{3}/2$.
 (b) Find all solutions between -180° and 180° of the equation $\cos \theta = -0.6$.

Solution

- (a) Sketch the graph of sine in the interval 0 to 2π . Sketch the horizontal line at height $\sqrt{3}/2$, and mark the crossing points. The θ -coordinates of these points are the required solutions of $\sin \theta = \sqrt{3}/2$.

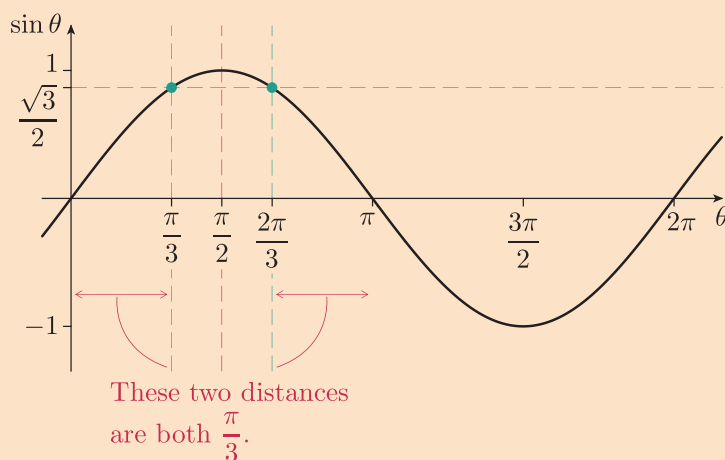


Find one solution using inverse sine.

One solution is

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

Use the symmetry of the graph to find the other solution.

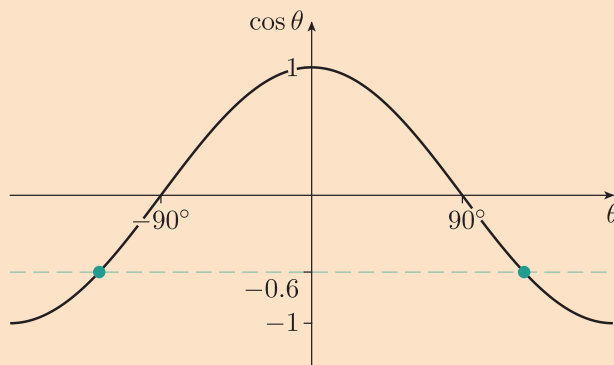


From the graph, the other solution is

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

So the solutions are $\pi/3$ and $2\pi/3$.

- (b) Sketch the graph of cosine in the interval -180° to 180° . Sketch the horizontal line at height -0.6 , and mark the crossing points. The θ -coordinates of these points are the required solutions of $\cos \theta = -0.6$.



- Find one solution using inverse cosine on a calculator.

One solution is

$$\theta = \cos^{-1}(-0.6) = 126.869\dots^\circ.$$

- Use the symmetry of the graph to find the other solution.

From the graph, the other solution is

$$\theta = -126.869 \dots^\circ.$$

So the solutions are -127° and 127° , to the nearest degree.

In the next activity, you can practise using the graph method demonstrated in Example 8 to solve simple trigonometric equations. As before, the graphs that you sketch don't need to be accurate: they just need to illustrate the symmetry properties of the trigonometric functions.

Activity 21 *Solving simple trigonometric equations using the graphs of the trigonometric functions*

- (a) Find all solutions between -180° and 180° of the equation $\tan \theta = 10$. Give your answers to the nearest degree.
- (b) Find all solutions between $-\pi$ and π of the equation $\sin \theta = -1/\sqrt{2}$.

Although it doesn't usually matter which of the two methods you use to solve a simple trigonometric equation, you can't use the ASTC method if the solutions of the equation are multiples of $\pi/2$. That is, you can't use it to solve any of the following seven 'special' trigonometric equations:

$$\begin{aligned}\sin \theta &= 1, & \cos \theta &= 1, \\ \sin \theta &= -1, & \cos \theta &= -1, \\ \sin \theta &= 0, & \cos \theta &= 0, & \tan \theta &= 0.\end{aligned}$$

You can solve these equations by using the graph method, or just by thinking about the definitions of sine, cosine and tangent. For example, suppose that you want to solve the equation $\sin \theta = 1$. This equation tells you that the point P associated with the angle θ has y -coordinate 1. So the solutions are $\theta = \pi/2$ and all angles obtained from $\pi/2$ by adding or subtracting integer multiples of 2π .

Activity 22 *Solving simple trigonometric equations whose solutions are multiples of $\pi/2$*

- (a) Find all solutions in the interval $[-\pi, \pi]$ of the equation $\sin \theta = 0$.
- (b) Find all solutions in the interval $[0, 4\pi]$ of the equation $\cos \theta = -1$.

3 Sine and cosine rules

In this section you'll see how to use trigonometry to find side lengths and angles in triangles that don't have a right angle. You'll also meet a useful trigonometric formula for the area of a triangle, and another formula that expresses the relationship between the gradient of a straight line and the angle that it makes with the x -axis.

3.1 The sine rule

Suppose that you want to work out the length of the side labelled b in the triangle in Figure 43. You have learned how to find side lengths of *right-angled* triangles, but the triangle here does not have a right angle, so the methods from Section 1 do not apply. Instead you must use a different method. The most convenient method is the *sine rule*, which is the subject of this subsection.

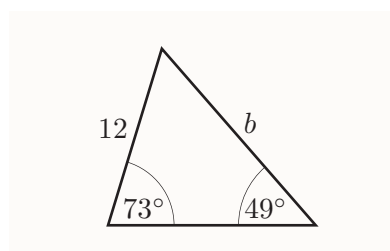


Figure 43 A triangle with an unknown side length b

The sine rule is most easily stated using the notation shown in Figure 44. Here the vertices of a triangle are labelled as A , B and C , and the sides are labelled as a , b and c . The labels are chosen in such a way that vertex A is opposite side a , vertex B is opposite side b , and vertex C is opposite side c . This is common notation for a triangle, which makes formulas easier to remember. It will be used throughout the rest of this unit.

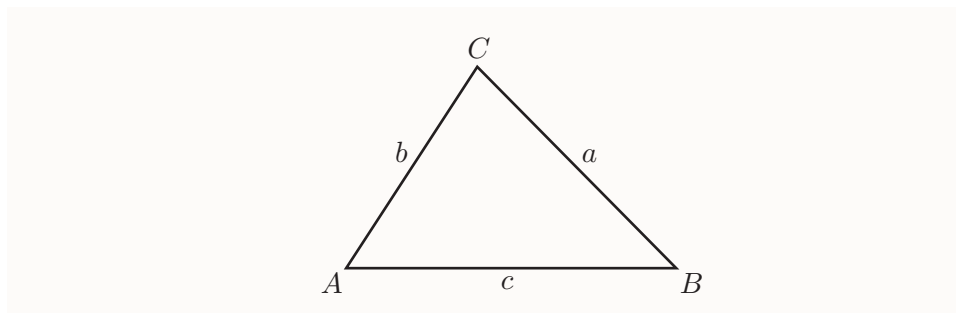


Figure 44 A triangle with vertices A , B and C and sides a , b and c

Note that the letters A , B and C will be used to represent both the vertices of the triangle *and* the angles at those vertices. For instance, we will write the sine of the angle at vertex A as $\sin A$.

Let's move on to discuss the sine rule. Figure 45 shows a triangle in which a dashed line of length h has been drawn from the vertex C , perpendicular to the opposite side c . Remember that two lines are **perpendicular** if they are at right angles to each other.

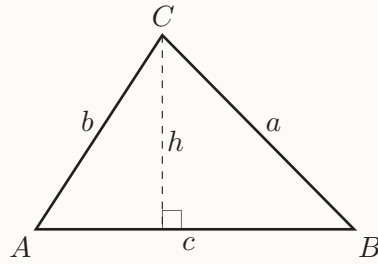


Figure 45 A triangle split into two right-angled triangles

The dashed line divides the triangle into two right-angled triangles. Because sine equals opposite over hypotenuse, you can see from the left-hand triangle that

$$\sin A = \frac{h}{b}, \quad \text{so} \quad h = b \sin A.$$

In the same way, using the right-hand triangle, you can see that

$$\sin B = \frac{h}{a}, \quad \text{so} \quad h = a \sin B.$$

Therefore

$$b \sin A = h = a \sin B.$$

This equation can be rearranged to give

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

In a similar way, using a dashed line drawn from the vertex A perpendicular to the opposite side a , you can show that

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

Combining this equation with the one above gives the following rule.

Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The angles of the triangle in Figure 45 are all acute. However the sine rule holds for any triangle, even if it has an obtuse angle (an angle that is greater than 90° but less than 180°). The justification of the sine rule for such a triangle is similar to the justification above, but the dashed line of length h may lie outside the triangle, as illustrated in Figure 46.

You can use the sine rule to find either unknown side lengths or unknown angles in triangles. Let's focus on unknown side lengths first. The following box describes when you can use the sine rule to find a side length of a triangle.

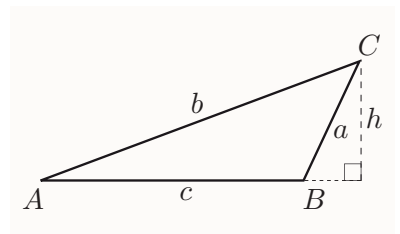


Figure 46 A triangle with an obtuse angle: the dashed line lies outside the triangle

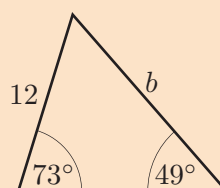
Using the sine rule to find a side length

You can use the sine rule to find an unknown side length of a triangle if you know

- the opposite angle
- another side length and its opposite angle.

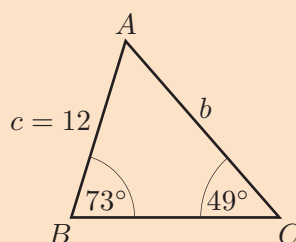
Example 9 Using the sine rule to find unknown side lengths

Find the side length b of the triangle below, giving your answer to one decimal place.



Solution

Label the vertices and the unknown side of the triangle.



The unknown side is b , and the known sides and angles are c , B and C . So use the sine rule in the form $\frac{b}{\sin B} = \frac{c}{\sin C}$.

By the sine rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C},$$

so

$$b = \frac{c \sin B}{\sin C} = \frac{12 \sin 73^\circ}{\sin 49^\circ} = 15.205 \dots = 15.2 \quad (\text{to 1 d.p.}).$$

Notice that in Example 9 the sine rule in the form

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

was rearranged to give

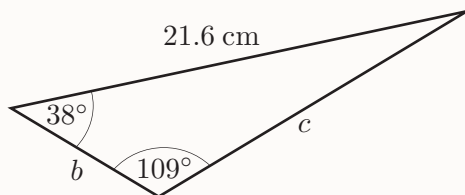
$$b = \frac{c \sin B}{\sin C},$$

before the values for B , C and c were substituted in. It's usually simpler to rearrange a trigonometric formula, like the sine rule, into the form that you want before you substitute in.

In the next activity, remember to set your calculator to degrees, rather than radians.

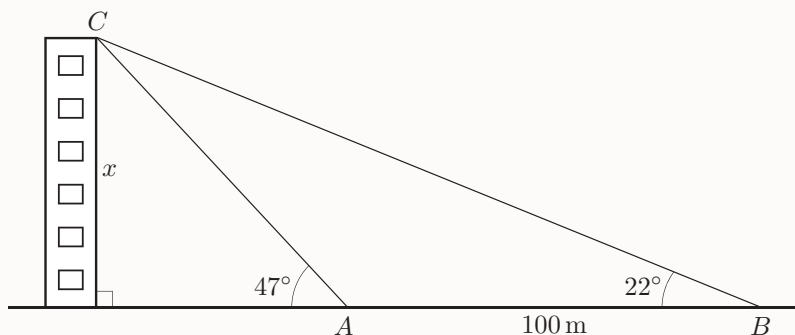
Activity 23 Using the sine rule to find an unknown side length

Find the side lengths b and c of the triangle below. Give your answers in centimetres to one decimal place.



Activity 24 Using the sine rule to find a distant height

The diagram below shows a building of unknown height x , in metres, and the values of two angles between the horizontal and a line to a point C on the top of the building, measured from points A and B that are 100 m apart. Use the sine rule to find the distance BC , and hence deduce the height of the building, both to the nearest metre.



Now let's look at how you can use the sine rule to find unknown *angles*.

Suppose that you know that two of the side lengths of a triangle are $a = 51$ and $b = 22$, and that one of the angles is $B = 23^\circ$, and you want to find the angle A . (Here we're using the usual notation for triangles, so side a is opposite angle A , and side b is opposite angle B .)

You can rearrange the sine rule in the form

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

to make $\sin A$ the subject:

$$\sin A = \frac{a \sin B}{b}.$$

You can then substitute in the values of a , b and B , to obtain

$$\sin A = \frac{a \sin B}{b} = \frac{51 \sin 23^\circ}{22} = 0.905 \dots$$

Because A is an angle of a triangle, it lies between 0° and 180° . However the equation $\sin A = 0.905 \dots$ has *two* solutions between 0° and 180° , namely 65° (to the nearest degree) and $180^\circ - 65^\circ = 115^\circ$ (to the nearest degree), as shown in Figure 47.

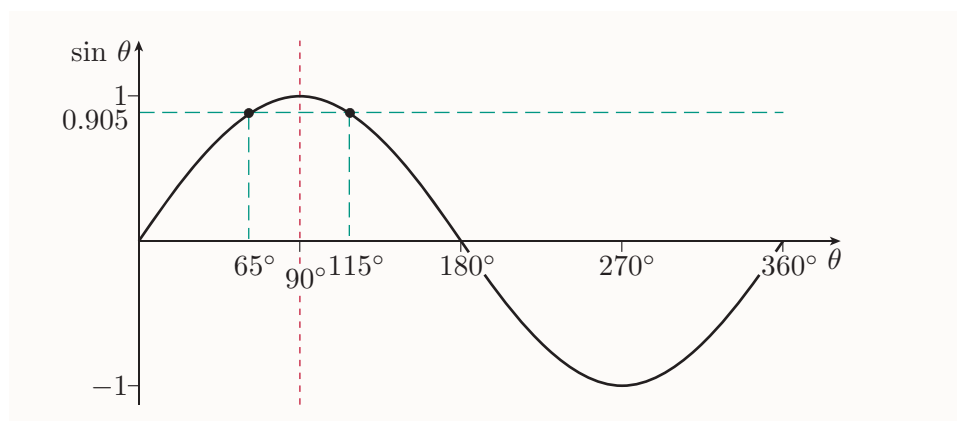


Figure 47 Two solutions to $\sin A = 0.905 \dots$

So the sine rule has provided two possible values for the angle A . Both of these possibilities can occur. You can draw a triangle with $a = 51$, $b = 22$, $B = 23^\circ$ and $A \approx 65^\circ$, and you can draw another triangle with $a = 51$, $b = 22$, $B = 23^\circ$ and $A \approx 115^\circ$, as shown in Figure 48.

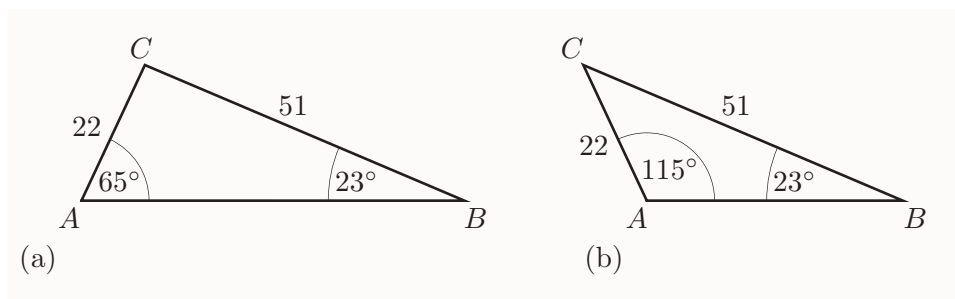


Figure 48 Two triangles that have $a = 51$, $b = 22$ and $B = 23^\circ$

So if you use the sine rule to find an unknown angle, then you may only be able to narrow down the answer to two possibilities: an acute angle, and an obtuse angle that is equal to 180° minus the acute angle. You can find the correct answer only if you have other information that tells you whether the unknown angle is acute or obtuse.

Activity 25 Using the sine rule to find an unknown acute angle

Suppose that a triangle has $A = 19.9^\circ$, $a = 7.6$ and $c = 16.4$. Suppose also that C is acute. Find the angle C , in degrees to one decimal place.

You have seen that when you use the sine rule to find an unknown angle you actually find two angles, one acute and one obtuse. The acute angle is always a valid solution, in the sense that you can always find a triangle with that acute angle and the other known sides and angles. In contrast, the obtuse angle is sometimes a valid solution, and sometimes it is not. For example, if adding the obtuse angle to the angle that you already know gives a total greater than 180° , then the obtuse angle is not a valid solution, because the sum of all three angles of a triangle is 180° .

You have seen an example where both the acute angle and the obtuse angle are valid solutions. Let's now look at an example in which the obtuse angle is not a valid solution.

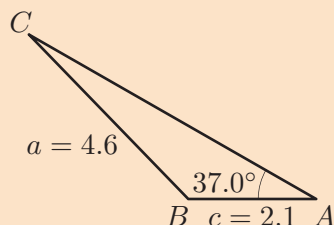
Example 10 Using the sine rule to find an unknown angle

Suppose that one angle in a triangle is $A = 37.0^\circ$, and two of its side lengths are $a = 4.6$ and $c = 2.1$. Find the angle C , in degrees to one decimal place.



Solution

Sketch the triangle (the sides and angles need not be accurately drawn), and label the known sides and angles.



The unknown angle is C , and the known sides and angles are a , c and A . So use the sine rule in the form $c/\sin C = a/\sin A$.

By the sine rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$

Rearranging gives

$$\sin C = \frac{c \sin A}{a} = \frac{2.1 \sin 37.0^\circ}{4.6} = 0.274 \dots$$

Find the acute and obtuse angles that satisfy $\sin C = 0.274 \dots$

The two possible solutions are

$$C = \sin^{-1}(0.274 \dots) = 15.946 \dots^\circ$$

and

$$C = 180^\circ - 15.946 \dots^\circ = 164.053 \dots^\circ.$$

Check whether the obtuse angle is possible.

If $C = 164.053 \dots^\circ$, then

$$A + C = 37.0^\circ + 164.053 \dots^\circ = 201.053 \dots^\circ.$$

This is impossible, because the sum of the angles $A + B + C$ must be 180° . Therefore $C = 15.9^\circ$, to one decimal place.

You may find it helpful to remember that, in a triangle, *larger angles are found opposite longer sides, and vice versa*. In Example 10, side a is longer than side c , so angle A must be larger than angle C . This gives you another way to check that angle C cannot be 164.1° , since angle C must be smaller than 37.0° , the value of angle A .

The following box summarises when you can use the sine rule to find an unknown angle.

Using the sine rule to find an angle

You can use the sine rule to find an unknown angle in a triangle if you know:

- the opposite side length
- another side length and its opposite angle.

Sometimes you also need to know whether the unknown angle is acute or obtuse.

Activity 26 Using the sine rule to find an unknown angle

One angle in a triangle is $B = 23.4^\circ$, and two side lengths are $b = 9.3$ and $c = 8.1$. Find the angle C , in degrees to one decimal place.

3.2 The cosine rule

There is another rule, known as the *cosine rule*, that you can use to find unknown side lengths and angles in triangles that are not necessarily right-angled triangles.

This rule can be used in some circumstances where the sine rule cannot be used. Take Figure 49, for example, in which the side length a is unknown. You cannot use the sine rule, because neither of the two known side lengths 13 and 15 is opposite the known angle 33° . Instead you can apply the cosine rule, as you'll see shortly.

Here's how the cosine rule is obtained. Once again, divide a triangle into two right-angled triangles by a dashed line of length h , as shown in Figure 50. The side of length c has been split into two parts: one part has length y , so the other part has length $c - y$.

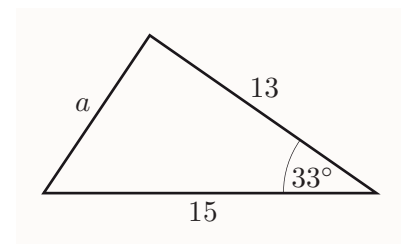


Figure 49 A triangle with an unknown side length a

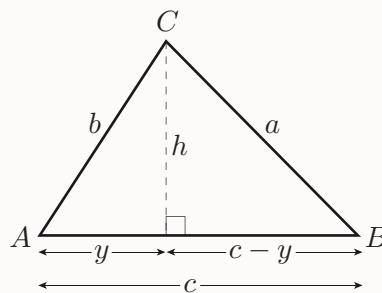


Figure 50 The dashed line divides the triangle into two right-angled triangles

Applying Pythagoras' theorem to each of the right-angled triangles gives

$$b^2 = y^2 + h^2,$$

$$a^2 = (c - y)^2 + h^2.$$

Expanding the brackets in the second of these equations gives

$$a^2 = c^2 - 2cy + y^2 + h^2.$$

Since $y^2 + h^2 = b^2$ it follows that

$$a^2 = c^2 - 2cy + b^2$$

$$= b^2 + c^2 - 2cy.$$

Next, using the right-angled triangle on the left in Figure 50 you find that

$$\cos A = \frac{y}{b}, \quad \text{so} \quad y = b \cos A.$$

Substituting this expression for y into $a^2 = b^2 + c^2 - 2cy$ gives

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This equation is one form of the cosine rule.

Cosine rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

There are two other equivalent forms of the cosine rule, that use other sides and angles:

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

The triangle in Figure 50 has only acute angles. However, like the sine rule, the cosine rule holds for any triangle, even if it has an obtuse angle. The justification of the cosine rule for such a triangle is similar to the justification above, but the dashed line of length h may lie outside the triangle, as illustrated in Figure 51.

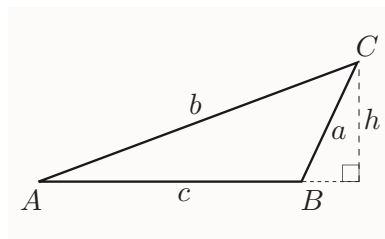


Figure 51 A triangle with an obtuse angle: the dashed line lies outside the triangle

Notice that if C is a right angle ($\pi/2$ or 90°), then $\cos C = 0$, and the last equation above becomes $a^2 + b^2 = c^2$, which is Pythagoras' theorem for a right-angled triangle. In this way you can see that the cosine rule is a more general version of Pythagoras' theorem. It applies to any triangle, whereas Pythagoras' theorem applies only to right-angled triangles.

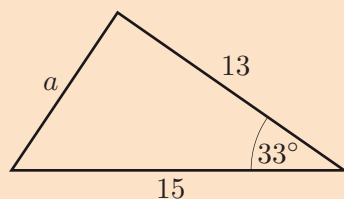
The following box describes when you can use the cosine rule to find an unknown side length.

Using the cosine rule to find a side length

You can use the cosine rule to find an unknown side length of a triangle if you know the other two side lengths and the angle between them.

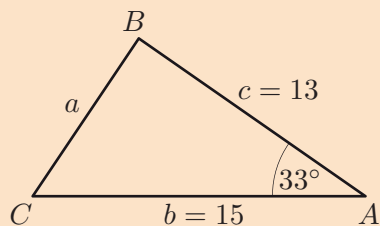
Example 11 Using the cosine rule to find an unknown side length

Find the length of the side a in the triangle below, to one decimal place.



Solution

Label the angles and sides of the triangle.



The known angle is A . So use the cosine rule in the form $a^2 = b^2 + c^2 - 2bc \cos A$.

By the cosine rule,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This gives

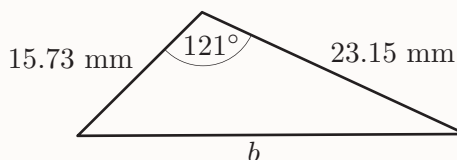
$$\begin{aligned} a &= \sqrt{b^2 + c^2 - 2bc \cos A} \\ &= \sqrt{15^2 + 13^2 - 2 \times 15 \times 13 \times \cos 33^\circ} \\ &= 8.180 \dots = 8.2 \quad (\text{to 1 d.p.}). \end{aligned}$$

In a calculation like the one at the end of the example above, it's best to type the whole numerical expression, including the square root, into your

calculator in order to avoid rounding errors. If instead you evaluate the expression under the square root sign first, then don't round it before taking the square root.

Activity 27 Using the cosine rule to find an unknown side length

Find the length of the side b in the triangle below, in millimetres to two decimal places.



You've seen that you can use the cosine rule to find an unknown side length of a triangle if you know the other two side lengths and the angle between them. In fact, there's another situation in which you can sometimes use the cosine rule to find an unknown side length, namely if you know the other two side lengths and the angle opposite one of them. However, in such circumstances it's simpler to start by using the sine rule. Note that there may be two possible answers.

Let's now use the cosine rule to find unknown angles. This is simpler than using the sine rule to find unknown angles, because you never obtain two possible angles. The box below describes when you can use the cosine rule to find an unknown angle.

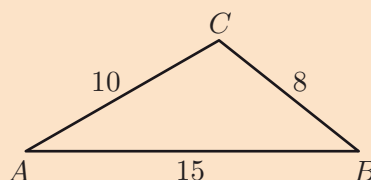
Using the cosine rule to find an angle

You can use the cosine rule to find an unknown angle if you know all three side lengths.



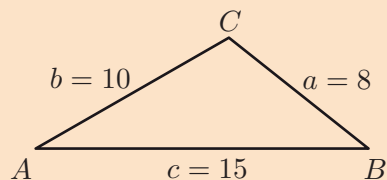
Example 12 Using the cosine rule to find an unknown angle

Use the cosine rule to find the angle C in the triangle below, in degrees to one decimal place.



Solution

Label the sides a , b and c .



You want to find angle C , so use the cosine rule in the form $c^2 = a^2 + b^2 - 2ab \cos C$. Rearrange it to obtain an expression for $\cos C$.

Rearranging $c^2 = a^2 + b^2 - 2ab \cos C$ gives

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Substitute in the values of a , b and c .

$$\cos C = \frac{8^2 + 10^2 - 15^2}{2 \times 8 \times 10}$$

Use inverse cosine to obtain a solution between 0° and 180° .

$$C = \cos^{-1} \left(\frac{8^2 + 10^2 - 15^2}{2 \times 8 \times 10} \right) = 112.4^\circ \quad (\text{to 1 d.p.}).$$

Here's an example for you to try.

Activity 28 *Using the cosine rule to find an unknown angle*

Find the angle opposite the side of length 3.6 m in a triangle with side lengths 1.5 m, 2.5 m and 3.6 m. Give your answer to the nearest degree.

You have now seen four methods of finding unknown side lengths and angles in a triangle:

- Pythagoras' theorem
- the trigonometric ratios sine, cosine and tangent
- the sine rule
- the cosine rule.

The process of choosing which of these methods to use can be summarised in the form of a decision tree, as shown in Figure 52.

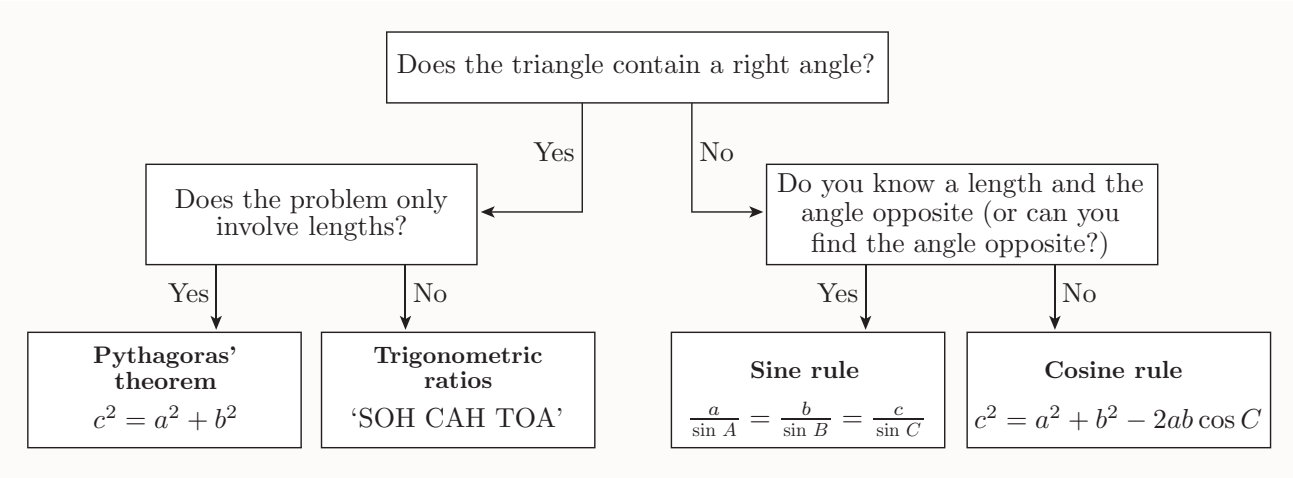


Figure 52 A decision tree showing methods of solving a triangle

All of these methods are used by surveyors when producing detailed plans for road and building construction, and they were used historically for creating maps, such as in the Great Trigonometric Survey (see page 21).

3.3 The area of a triangle

In this short subsection you’ll meet a useful formula for the area of a triangle, which involves the sine of one of the angles of the triangle. Our starting point is the more familiar formula below.

Area of a triangle

For a triangle,

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}.$$

Remember that when you use this formula, the **base** of the triangle is one of its sides. Any side will do; different sides just lead to different calculations. The **height** of the triangle is the perpendicular distance from the base to the opposite vertex. Again, the height depends on the choice of base. Usually you draw the base horizontally and the height vertically, as shown in Figure 53.

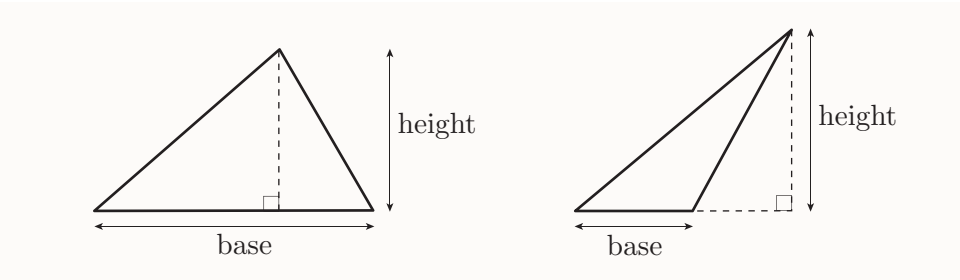


Figure 53 The bases and heights of two triangles

Let's use the formula above to obtain another formula for the area of a triangle, in terms of the lengths of two sides and the angle between these sides. Consider the triangle in Figure 54. The sides are labelled a and b , and the angle between them is labelled θ .

First draw a dashed line of length h perpendicular to the base b , as shown in Figure 55. Then the area of the triangle is $\frac{1}{2}bh$. From the right-angled triangle on the left of Figure 55,

$$\sin \theta = \frac{h}{a}, \quad \text{so} \quad h = a \sin \theta.$$

Substituting this expression for h into the formula $\frac{1}{2}bh$ gives

$$\text{area} = \frac{1}{2}bh = \frac{1}{2}b \times a \sin \theta = \frac{1}{2}ab \sin \theta.$$

This is the new formula for the area of a triangle.

Area of a triangle

For a triangle with an angle θ between two sides of lengths a and b ,

$$\text{area} = \frac{1}{2}ab \sin \theta.$$

In this derivation of the formula for the area of a triangle, the angle θ was acute. If the angle θ is obtuse, then the area formula still works, but its justification is slightly different because the dashed line lies outside the triangle; the details are omitted.

You can use this area formula in the following activity. The activity involves a *regular* pentagon, which is a pentagon with equal angles and equal side lengths.

Activity 29 Finding the area of a pentagon

By dividing the regular pentagon below into triangles, find its area in m^2 , to one decimal place.

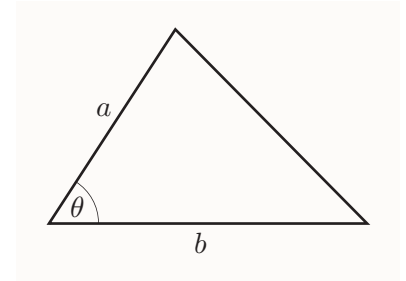
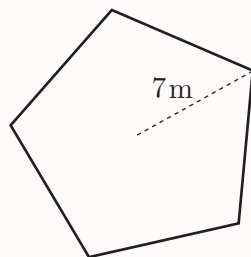


Figure 54 A triangle with an angle θ between two sides of lengths a and b

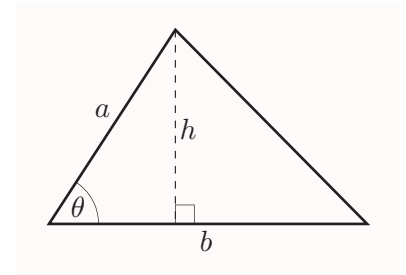
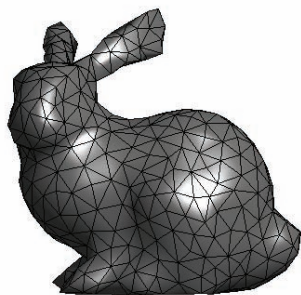


Figure 55 The triangle in Figure 54 with its height labelled as h



A triangulated rabbit

Triangulation

In mathematics, the word *triangulate* means to divide up into triangles. Any polygon can be triangulated, as illustrated in Figure 56. (A *polygon* is a flat shape whose boundary consists of line segments.)

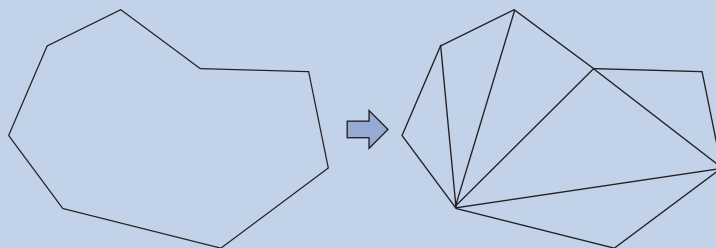


Figure 56 Triangulating a polygon

You can calculate the area of a polygon by triangulating it, calculating the areas of each of the individual triangles, and then adding these to give the total area.

You can also triangulate surfaces in three dimensions. This technique is particularly useful for computer programmers who work with graphics. After triangulating a surface, a programmer can use the triangulation to animate the object.

The next activity asks you to think about other formulas for the area of a triangle. In each part, you should try to think of a mathematical reason why there is or isn't a formula – you don't need to actually find any formulas (which would be difficult).

Activity 30 Thinking about other formulas for the area of a triangle

- Do you think that there is a formula for the area of a triangle in terms of only the angles A , B and C ?
- Do you think that there is a formula for the area of a triangle in terms of only the side lengths a , b and c ?

3.4 The angle of inclination of a line

In Unit 2 you saw that every straight line (that isn't vertical) has an equation of the form $y = mx + c$. The number m is the *gradient* of the line: if it is positive, then the line slopes up from left to right, and if it is negative, then the line slopes down from left to right. The number c is the y -intercept. In this subsection you'll learn how the trigonometric function tangent can be helpful in describing the gradient of a straight line.

The **angle of inclination** of a straight line is the angle that it makes with the x -axis, measured anticlockwise from the positive direction of the x -axis, when the line is drawn on axes *with equal scales*. A line with a positive gradient has an angle of inclination between 0 and $\pi/2$, as illustrated in Figure 57(a). A line with a negative gradient has an angle of inclination between $\pi/2$ and π , as illustrated in Figure 57(b).

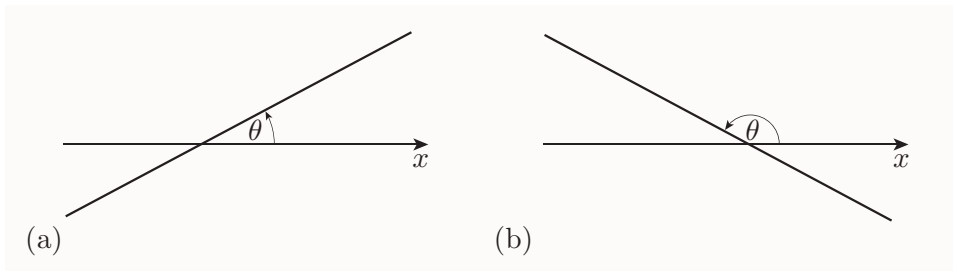


Figure 57 The angle of inclination θ of (a) a line with a positive gradient
(b) a line with a negative gradient

To understand the relationship between the gradient of a line and its angle of inclination, consider any angle θ between 0 and π (but not equal to $\pi/2$), and its associated point $P(x, y)$ on the unit circle, as illustrated in Figure 58. The angle θ is the angle of inclination of the line OP and, since $\theta \neq \pi/2$, the line OP is not vertical, and hence has a gradient.

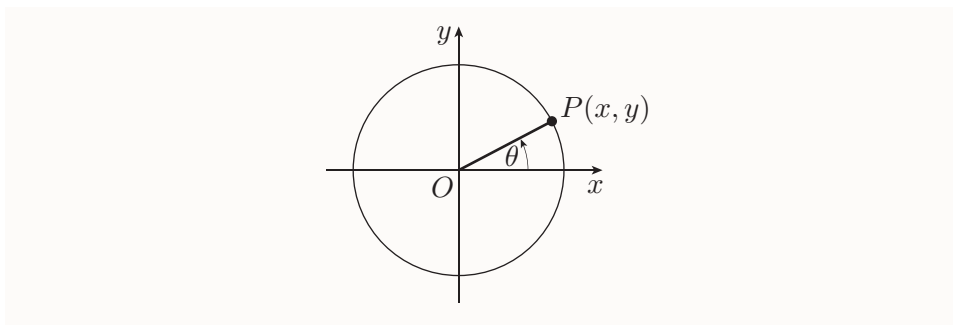


Figure 58 An angle θ and its associated point P on the unit circle

You saw earlier that

$$\tan \theta = \frac{y}{x}.$$

Also, by the usual formula for gradient,

$$\text{gradient of line } OP = \frac{y - 0}{x - 0} = \frac{y}{x}.$$

So we have the following useful fact.

Gradient and angle of inclination of a straight line

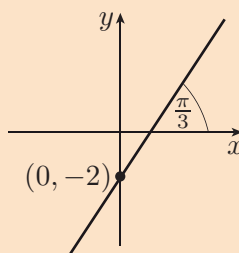
For any non-vertical straight line with angle of inclination θ ,

$$\text{gradient} = \tan \theta.$$

Remember that the angle of inclination is measured when the line is drawn on axes *with equal scales*.

Example 13 Finding the equation of a line with a given angle of inclination

Without using a calculator, find the equation of the straight line shown below.



Solution

Write down the general equation of a line.

The equation of the line is $y = mx + c$, where m is the gradient and c is the y -intercept.

Find the y -intercept.

The y -intercept is -2 , since the line passes through the point $(0, -2)$.

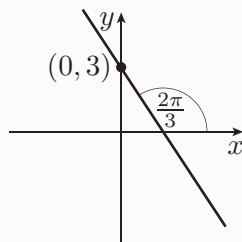
Use the formula for gradient in the box above.

The gradient is $\tan(\pi/3) = \sqrt{3}$. So the equation of the line is

$$y = \sqrt{3}x - 2.$$

Activity 31 Finding the equation of a line with a given angle of inclination

Without using a calculator, find the equation of the straight line shown below.



4 Further trigonometric identities

In this final section you'll see some of the many identities that involve the sines, cosines and tangents of angles. These identities are useful in many areas of mathematics. For example, you'll use some of them in the units on calculus and complex numbers later in the module.

There's no need to memorise all the trigonometric identities that you'll meet in this section: it's more important that you become familiar with using them, and you'll achieve this through practice in applying them. They're all listed in the *Handbook*. However, it's worth memorising the two simple identities in Subsection 4.1, as they're used very frequently.

4.1 Two simple trigonometric identities

You have met the two trigonometric identities in this subsection already: you saw in Subsection 1.6 that they're true for *acute angles*. Here you'll see that they're true for *all* angles for which the expressions in them are defined.

For the first identity, remember that $\cos \theta$ and $\sin \theta$ are the x - and y -coordinates of the point P on the unit circle associated with the angle θ , and $\tan \theta$ is the y -coordinate divided by the x -coordinate (when the x -coordinate is not zero). This gives the identity below.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$



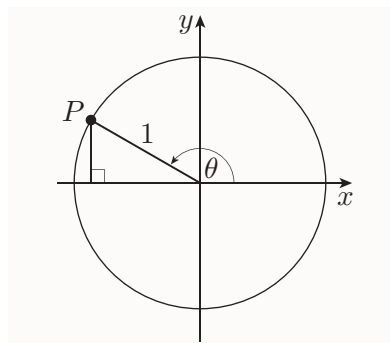


Figure 59 A point P on the unit circle

For the second identity, again consider an angle θ and its associated point P on the unit circle. If P doesn't lie on one of the axes, then you can draw a line from P perpendicular to the x -axis to form a right-angled triangle. Let's consider the case in which P lies in the second quadrant, as illustrated in Figure 59. The length of the vertical side of the triangle is the y -coordinate of P , which is $\sin \theta$. The length of the horizontal side is the negative of the x -coordinate of P (since the x -coordinate is negative), so it is $-\cos \theta$. Hence, by Pythagoras' theorem,

$$(\sin \theta)^2 + (-\cos \theta)^2 = 1^2,$$

which simplifies to

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

You can see that a similar argument holds if P is in any of the other three quadrants, so the equation above is true for all the corresponding values of θ . The equation is also true if P lies on one of the axes: for example, if $\theta = \pi/2$, then P is at the top of the circle, so $\sin \theta = 1$ and $\cos \theta = 0$. So the equation is an identity.

As mentioned earlier, the expressions $(\sin \theta)^2$ and $(\cos \theta)^2$ are usually written as $\sin^2 \theta$ and $\cos^2 \theta$, so the identity above is usually written as follows.

$$\sin^2 \theta + \cos^2 \theta = 1$$

Activity 32 Writing sine in terms of cosine

Show that

$$\sin \theta = \begin{cases} \sqrt{1 - \cos^2 \theta} & \text{if } \sin \theta \text{ is positive,} \\ -\sqrt{1 - \cos^2 \theta} & \text{if } \sin \theta \text{ is negative.} \end{cases}$$



Claudius Ptolemy
(c. AD 90–168)

Many of the trigonometric identities in this section were known, in somewhat different forms, to the ancient Egyptian mathematician, astronomer and geographer Claudius Ptolemy. Ptolemy, who was of Greek descent, authored the influential book on astronomy *Almagest* – the name derives from the Arabic word 'al-majisti', which in turn derives from the Greek word for 'greatest'. The book is a rich source of trigonometry, some of which was based on the work (now lost) of the ancient Greek mathematician Hipparchus (see page 15).

4.2 Trigonometric identities from the graphs of sine, cosine and tangent

In this subsection you'll see how to use the symmetry properties of the graphs of the sine, cosine and tangent functions (Figure 60) to deduce some trigonometric identities.

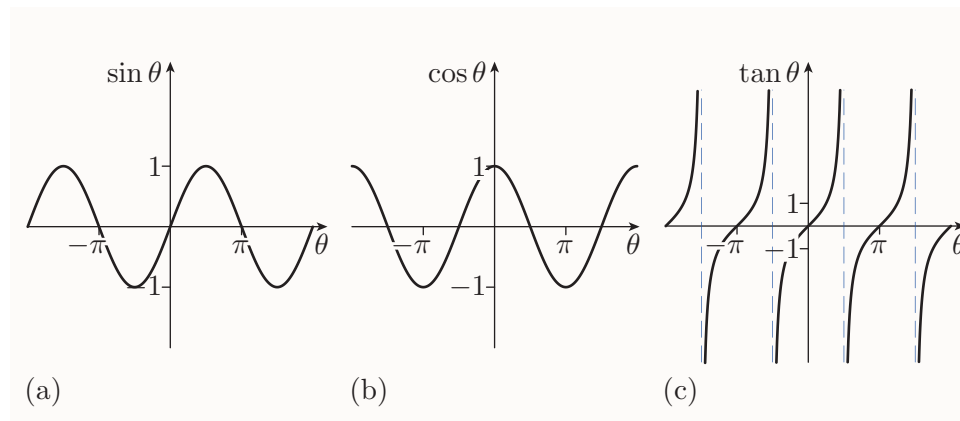


Figure 60 The graphs of the sine, cosine and tangent functions

As you learned earlier, the trigonometric functions are periodic: sine and cosine have period 2π , and tangent has period π . These facts give the identities below.

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\cos(\theta + 2\pi) = \cos \theta$$

$$\tan(\theta + \pi) = \tan \theta$$

The next set of identities comes from other symmetry properties of the graphs of the trigonometric functions. The vertical axis is a line of mirror symmetry for the graph of cosine, which means that $\cos \theta$ and $\cos(-\theta)$ are equal, for every angle θ . This gives the identity $\cos(-\theta) = \cos \theta$. However, the sine of $-\theta$ is equal to the *negative* of the sine of θ , as you can see from Figure 60(a). That is, $\sin(-\theta) = -\sin \theta$, for every angle θ . There is a similar identity for tangent.

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

For the final pair of identities in this subsection, notice from Figure 60 that if you translate the graph of cosine to the left by $\pi/2$ and then reflect it in the vertical axis, you obtain the graph of sine. In other words, from what

you saw about translating and reflecting graphs in Unit 3, the graph of the function

$$g(\theta) = \cos\left(-\theta + \frac{\pi}{2}\right)$$

is the same as the graph of the sine function. This in turn tells you that the equation

$$\cos\left(-\theta + \frac{\pi}{2}\right) = \sin \theta$$

is an identity.

Notice also that if you translate the graph of sine to the left by $\pi/2$ and then reflect it in the vertical axis, you obtain the graph of cosine. This gives the identity

$$\sin\left(-\theta + \frac{\pi}{2}\right) = \cos \theta.$$

These two identities are stated in a slightly neater form below.

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

You met these identities for acute angles earlier in the unit.

You can use the symmetries of the graphs of the sine, cosine and tangent functions to obtain many more trigonometric identities.

4.3 Cosecant, secant and cotangent

In Section 1, the sine, cosine and tangent of an acute angle θ were defined as *ratios* of the sides of a right-angled triangle that has θ as one of its acute angles (Figure 61). Specifically, they were defined by

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \tan \theta = \frac{\text{opp}}{\text{adj}}.$$

There are three further ratios of the sides of such a triangle. These ratios are known as the **cosecant**, **secant** and **cotangent** of θ (pronounced as ‘co-see-cant’, ‘see-cant’ and ‘co-tan-gent’). They are written as $\text{cosec } \theta$, $\text{sec } \theta$ and $\text{cot } \theta$, and defined as follows:

$$\text{cosec } \theta = \frac{\text{hyp}}{\text{opp}}, \quad \text{sec } \theta = \frac{\text{hyp}}{\text{adj}} \quad \text{and} \quad \text{cot } \theta = \frac{\text{adj}}{\text{opp}}.$$

Cosec, sec and cot are pronounced as ‘co-seck’, ‘seck’ and ‘cot’.

From these definitions you can see that the equations below hold. We use these equations as the definitions of cosecant, secant and cotangent for *any* angle θ – not just for acute angles.

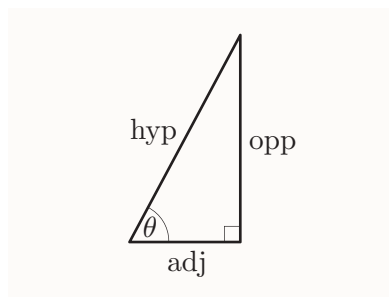


Figure 61 A right-angled triangle with acute angle θ

Cosecant, secant and cotangent

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad (\text{provided } \sin \theta \neq 0)$$

$$\sec \theta = \frac{1}{\cos \theta} \quad (\text{provided } \cos \theta \neq 0)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (\text{provided } \sin \theta \neq 0)$$

Since you have already seen that

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (\text{provided } \cos \theta \neq 0),$$

the third equation above gives the following equation.

$$\cot \theta = \frac{1}{\tan \theta} \quad (\text{provided } \sin \theta \neq 0 \text{ and } \cos \theta \neq 0)$$

So, essentially, the cosecant, secant and cotangent functions are the reciprocals of the sine, cosine and tangent functions, respectively. Since the reciprocal of a periodic function has the same period as the original function, the cosecant and secant functions have period 2π and the cotangent function has period π .

Sine, cosine and tangent, together with cosecant, secant and cotangent, form the complete set of six trigonometric functions. The graphs of cosecant, secant and cotangent are shown in Figure 62.

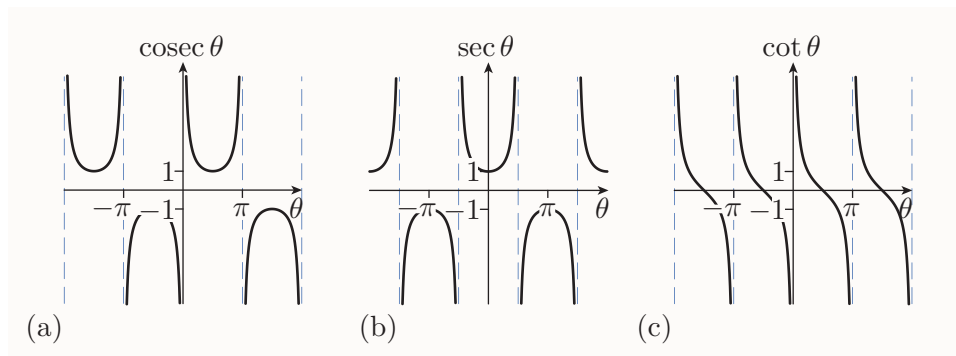


Figure 62 The graphs of the cosecant, secant and cotangent functions

You can work out values of cosecant, secant and cotangent for some special angles by using the values of sine, cosine and tangent for these angles, as illustrated in the next example.

Example 14 *Finding values of cosecant, secant and cotangent*

Find the exact values of $\operatorname{cosec}(\pi/6)$, $\sec(\pi/6)$ and $\cot(\pi/6)$, without using a calculator.

Solution

 Use the values of $\sin(\pi/6)$ and $\cos(\pi/6)$ from the special angles table. 

$$\operatorname{cosec} \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{1/2} = 2$$

$$\sec \frac{\pi}{6} = \frac{1}{\cos \frac{\pi}{6}} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

$$\cot \frac{\pi}{6} = \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Activity 33 *Finding values of cosecant, secant and cotangent*

Find the value of each of the following cosecants, secants and cotangents (or state that it is undefined), without using a calculator.

- (a) $\operatorname{cosec} 0$, $\sec 0$ and $\cot 0$
- (b) $\operatorname{cosec} \frac{\pi}{2}$, $\sec \frac{\pi}{2}$ and $\cot \frac{\pi}{2}$
- (c) $\operatorname{cosec} \frac{\pi}{4}$, $\sec \frac{\pi}{4}$ and $\cot \frac{\pi}{4}$

You can use the identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

to obtain two new identities involving cosecant, secant and cotangent. Dividing both sides of the identity above by $\cos^2 \theta$ gives

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}; \quad \text{that is,} \quad \tan^2 \theta + 1 = \sec^2 \theta.$$

Dividing both sides of the original identity by $\sin^2 \theta$ instead of $\cos^2 \theta$ gives another new identity. Both of these new identities are sometimes useful, and are stated in the following box.

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

4.4 Angle sum and angle difference identities

In this subsection and the next you'll meet some further trigonometric identities. These allow you to obtain the exact values of the sine, cosine and tangent of many more angles than the few special angles that you've seen so far, and they're also useful in other contexts, such as calculus.

Let's begin with an identity that expresses $\sin(A + B)$ in terms of $\sin A$, $\sin B$, $\cos A$ and $\cos B$. This identity is known as the *angle sum identity for sine*. The letters A and B represent any angles (they don't have to be two angles from a triangle).

Here's how the identity can be obtained in the case where A and B are acute angles. Suppose that you have a right-angled triangle with an acute angle A , an adjacent side of length 1, and hypotenuse of length p , as shown on the left of Figure 63. You also have another right-angled triangle with an acute angle B , an adjacent side of length 1, and hypotenuse of length q , as shown on the right of Figure 63.

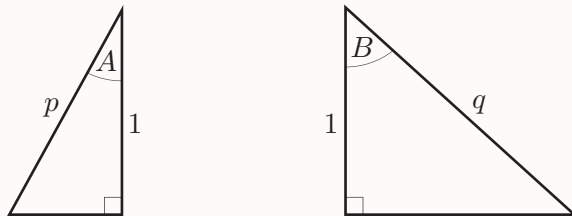


Figure 63 Two right-angled triangles

If you stick these two triangles together along their sides of length 1, then you obtain the larger triangle shown in Figure 64.

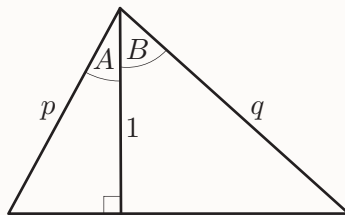


Figure 64 A triangle divided into two right-angled triangles

Let's now apply the formula for the area of a triangle that you met earlier. Remember that it tells you that the area of a triangle with an angle θ between sides of lengths a and b (as shown in Figure 65) is $\frac{1}{2}ab \sin \theta$. We'll

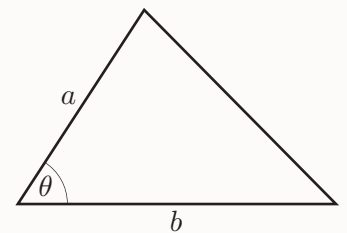


Figure 65 A triangle with an angle θ between two sides of lengths a and b

apply this formula three times to the triangle in Figure 64, as follows. The area of the right-angled triangle on the left is $\frac{1}{2}p \sin A$, the area of the right-angled triangle on the right is $\frac{1}{2}q \sin B$, and the area of the large triangle is $\frac{1}{2}pq \sin(A + B)$. But the area of the large triangle is also equal to the sum of the areas of the two right-angled triangles. That is,

$$\frac{1}{2}pq \sin(A + B) = \frac{1}{2}p \sin A + \frac{1}{2}q \sin B.$$

You can multiply both sides of this equation by 2 and then divide both sides by pq to obtain

$$\sin(A + B) = \frac{1}{q} \sin A + \frac{1}{p} \sin B.$$

The right-angled triangle on the left of Figure 64 gives

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{1}{p},$$

and the right-angled triangle on the right of Figure 64 gives

$$\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{1}{q}.$$

Together the last three equations give

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

This is the angle sum identity for sine. Although the justification above applies only when the angles A and B are acute, the identity holds for *all* angles A and B . This can be proved by considering various types of angles A and B , and using symmetry properties of sine and cosine to deduce the identity for each of these types from the case of acute angles.

You can obtain a similar identity for cosine. Since the angle sum identity for sine is true for any angles A and B , it remains true if you replace A by $\pi/2 - A$, and B by $-B$, where again A and B are any angles. This gives

$$\sin\left(\frac{\pi}{2} - A - B\right) = \sin\left(\frac{\pi}{2} - A\right) \cos(-B) + \cos\left(\frac{\pi}{2} - A\right) \sin(-B).$$

Note that $\sin(\pi/2 - A - B) = \sin(\pi/2 - (A + B))$. Earlier on, in Subsection 4.2, you met the identities

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta, & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta, \\ \sin(-\theta) &= -\sin \theta, & \cos(-\theta) &= \cos \theta. \end{aligned}$$

You can apply these identities as follows:

$$\underbrace{\sin\left(\frac{\pi}{2} - (A + B)\right)}_{\cos(A+B)} = \underbrace{\sin\left(\frac{\pi}{2} - A\right)}_{\cos A} \underbrace{\cos(-B)}_{\cos B} + \underbrace{\cos\left(\frac{\pi}{2} - A\right)}_{\sin A} \underbrace{\sin(-B)}_{-\sin B}.$$

This gives the angle sum identity for cosine:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

The angle sum identities for sine and cosine are stated again in the following box, for convenience.

Angle sum identities for sine and cosine

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

If you divide the angle sum identity for sine by the angle sum identity for cosine, then you obtain

$$\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing both the numerator and denominator of the fraction on the right-hand side by $\cos A \cos B$ gives

$$\tan(A + B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

This is the angle sum identity for tangent.

Angle sum identity for tangent

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

You have now seen all three angle sum identities. If you replace B by $-B$ in each of the three angle sum identities, and use the identities

$$\sin(-B) = -\sin B, \quad \cos(-B) = \cos B \quad \text{and} \quad \tan(-B) = -\tan B,$$

then you obtain the following three *angle difference identities*.

Angle difference identities

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Let's now look at how you can use the angle sum and angle difference identities to find the exact values of the sine, cosine and tangent of some more angles. If you know the exact values of the sine, cosine and tangent of two particular angles A and B (for example, from the special angles table), then you can apply the angle sum and angle difference identities to find the exact values of the sine, cosine and tangent of the angles $A + B$ and $A - B$. Here's an example.

**Example 15** *Finding an exact value of cosine*

Find the exact value of $\cos(7\pi/12)$, without using a calculator.

Solution

Find angles A and B such that you know the exact values of $\sin A$, $\sin B$, $\cos A$ and $\cos B$, and one of the equations $A + B = 7\pi/12$ or $A - B = 7\pi/12$ holds.

Let $A = \pi/3$ and $B = \pi/4$. Then

$$A + B = \frac{\pi}{3} + \frac{\pi}{4} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{7\pi}{12}.$$

Use the angle sum identity for cosine.

The identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$ gives

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ &= \cos\frac{\pi}{3} \cos\frac{\pi}{4} - \sin\frac{\pi}{3} \sin\frac{\pi}{4}\end{aligned}$$

Use the special angles table.

$$\begin{aligned}&= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} \\ &= \frac{1 - \sqrt{3}}{2\sqrt{2}}\end{aligned}$$

For a neater answer, rationalise the denominator. You saw how to do this in Unit 1. Here you multiply both the numerator and denominator by $\sqrt{2}$.

$$\begin{aligned}&= \frac{\sqrt{2}(1 - \sqrt{3})}{2\sqrt{2}\sqrt{2}} \\ &= \frac{\sqrt{2} - \sqrt{6}}{2 \times 2} \\ &= \frac{1}{4}(\sqrt{2} - \sqrt{6}).\end{aligned}$$

Activity 34 Finding exact values of sine, cosine and tangent

Find the exact values of the following, without using a calculator.

(a) $\sin\left(\frac{7\pi}{12}\right)$ (b) $\tan\left(\frac{\pi}{12}\right)$

Richard of Wallingford

Richard of Wallingford (1292–1336) was a priest and a scholar at Oxford University. He wrote a book on trigonometry and its applications to astronomy, called *Quadripartitum de sinibus demonstratis* (Demonstrations of sines, in four parts). The first part of the book is on trigonometric identities, which he used to calculate values of sine and cosine.



Richard of Wallingford, working in cramped conditions

4.5 Double-angle and half-angle identities

This subsection is about two more sets of trigonometric identities, the *double-angle identities* and the *half-angle identities*. The double-angle identities are obtained from the angle sum identities for sine, cosine and tangent by choosing A and B to be the same angle, say θ .

The angle sum identity for sine is

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Taking $A = \theta$ and $B = \theta$ gives

$$\sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta;$$

that is,

$$\sin(2\theta) = 2 \sin \theta \cos \theta.$$

Next, the angle sum identity for cosine is

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

Taking $A = \theta$ and $B = \theta$ gives

$$\cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta;$$

that is,

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta.$$

Finally, the angle sum identity for tangent is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Taking $A = \theta$ and $B = \theta$ gives

$$\tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta};$$

that is,

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

These three identities are stated again below, for convenience.

Double-angle identities

$$\begin{aligned}\sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

There are two alternative forms of the cosine double-angle identity. To obtain these, recall that $\sin^2 \theta + \cos^2 \theta = 1$, from which it follows that $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos^2 \theta = 1 - \sin^2 \theta$. Therefore

$$\begin{aligned}\cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= (1 - \sin^2 \theta) - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta,\end{aligned}$$

and

$$\begin{aligned}\cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= \cos^2 \theta - (1 - \cos^2 \theta) \\ &= 2 \cos^2 \theta - 1.\end{aligned}$$

Alternative double-angle identities for cosine

$$\begin{aligned}\cos(2\theta) &= 1 - 2 \sin^2 \theta \\ \cos(2\theta) &= 2 \cos^2 \theta - 1\end{aligned}$$

These alternative double-angle identities for cosine can be rearranged into the forms below. In these forms they are known as the *half-angle identities*.

Half-angle identities

$$\begin{aligned}\sin^2 \theta &= \frac{1}{2}(1 - \cos(2\theta)) \\ \cos^2 \theta &= \frac{1}{2}(1 + \cos(2\theta))\end{aligned}$$

The description ‘half-angle’ refers to the fact that these identities are sometimes stated with θ and 2θ replaced by $\frac{1}{2}\theta$ and θ , respectively, as follows:

$$\sin^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 - \cos\theta)$$

$$\cos^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 + \cos\theta).$$

Like the angle sum and angle difference identities in the last subsection, the half-angle identities can be used to provide the exact values of the sines, cosines and tangents of some further angles. Specifically, if you know the exact value of the cosine of a particular angle, then you can use the half-angle identities to find the exact values of the sine and cosine (and hence tangent) of half of that angle. The exact values that you obtain can be quite complicated, however, as they often contain square roots within square roots. Here’s an example.

Example 16 *Using a half-angle identity to find an exact value of sine*

Find the exact value of $\sin(\pi/8)$, without using a calculator.

Solution

 The angle $\pi/8$ is half of $\pi/4$, which is one of the angles in the special angles table. So use a half-angle identity. 

The half-angle identity for sine is

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)).$$

 Substitute in $\theta = \pi/8$. 

So

$$\sin^2\frac{\pi}{8} = \frac{1}{2}\left(1 - \cos\frac{\pi}{4}\right)$$

 Use the fact that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. 

$$\begin{aligned} &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{2} - 1}{2\sqrt{2}} \end{aligned}$$



 Rationalise the denominator. 

$$\begin{aligned}
 &= \frac{\sqrt{2}(\sqrt{2} - 1)}{2\sqrt{2}\sqrt{2}} \\
 &= \frac{2 - \sqrt{2}}{4}.
 \end{aligned}$$

 To obtain the required answer, take square roots of both sides. 

The angle $\pi/8$ is acute, so $\sin(\pi/8)$ is positive, and hence

$$\sin \frac{\pi}{8} = \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$

Here is a similar example for you to try.

Activity 35 *Using a half-angle identity to find an exact value of cosine*

Use a half-angle identity to find the exact value of $\cos(\pi/8)$, without using a calculator.

As pointed out in Section 1, once you've found the exact value of the sine, cosine or tangent of a particular acute angle, you can find the exact values of the other two trigonometric ratios for the angle by sketching a suitable right-angled triangle, and using Pythagoras' theorem and the definitions of sine, cosine and tangent. (Alternatively you can find them without sketching a triangle by using the identities $\sin^2 \theta + \cos^2 \theta = 1$ and $\tan \theta = \sin \theta / \cos \theta$.) So you can use the answer to Example 16 to obtain the answer to Activity 35, instead of using a half-angle identity. You can also use it to obtain the exact value of $\tan(\pi/8)$. However, the answer you obtain this way is

$$\tan \frac{\pi}{8} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}},$$

which is quite complicated. You can obtain a simpler form of the answer by finding it in a different way, using a trigonometric identity, as you'll see in the next activity. This illustrates that there are often various ways of working with trigonometric ratios.

Activity 36 *Using a half-angle identity to find an exact value of tangent*

Use the double-angle formula for tangent with $2\theta = \pi/4$ to calculate an exact value for $\tan(\pi/8)$.

Hint: if you write $t = \tan(\pi/8)$, then you can use the double-angle formula for tangent to give you a quadratic equation for t .

At the end of the solution to Activity 36 there is an explanation of why the simpler form of the exact value of $\tan(\pi/8)$ is equal to the more complicated form given above.

4.6 Yet more trigonometric identities

The angle sum and angle difference identities, and the double-angle identities, can be used to deduce many new trigonometric identities, some of which are useful in other subjects such as calculus.

For example, consider the angle sum and angle difference identities for sine:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Adding these gives

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

This identity enables you to convert a product of a sine and a cosine on the right-hand side into the sum of two sines on the left-hand side, which is sometimes useful. There are similar identities, which can be obtained in a similar way, that convert products of two cosines and products of two sines into sums or differences of cosines.

The new identity above can be rearranged to give a slightly different identity that has some useful applications. Suppose that you let

$$C = A + B \quad \text{and} \quad D = A - B.$$

Solving these equations for A and B , by first adding them and then subtracting them, gives

$$A = \frac{C + D}{2} \quad \text{and} \quad B = \frac{C - D}{2}.$$

Substituting into the identity above gives

$$\sin C + \sin D = 2 \sin \left(\frac{C + D}{2} \right) \cos \left(\frac{C - D}{2} \right).$$

This identity relates a sum of two sines on the left-hand side to a product of a sine and a cosine on the right-hand side. There are similar identities, which can be obtained in a similar way, for a difference of two sines, and

for a sum and difference of two cosines. These are listed in the following box, with the variables C and D replaced by the usual variables A and B .

$$\begin{aligned}\sin A + \sin B &= 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \\ \sin A - \sin B &= 2 \sin \left(\frac{A-B}{2} \right) \cos \left(\frac{A+B}{2} \right) \\ \cos A + \cos B &= 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \\ \cos A - \cos B &= -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)\end{aligned}$$

These identities can be used whenever you find it useful to change a sum or difference of sines or cosines into a product of trigonometric functions, or vice versa. As with other trigonometric identities, there is no need for you to memorise them.

You can use trigonometric identities to explain an interesting effect that's obtained when two musical notes with similar frequencies are played at the same time.

Consider the graphs of the functions $f(t) = \sin(20t)$ and $g(t) = \sin(22t)$, which are shown in Figure 66. They represent waves, such as sound waves, that are almost the same but have slightly different periods.

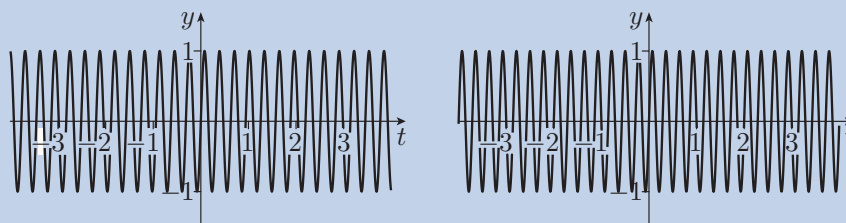


Figure 66 The graphs of (a) $f(t) = \sin(20t)$ (b) $g(t) = \sin(22t)$

Now consider what happens when you add these two functions. You obtain the function

$$h(t) = \sin(20t) + \sin(22t),$$

whose graph is shown in Figure 67.

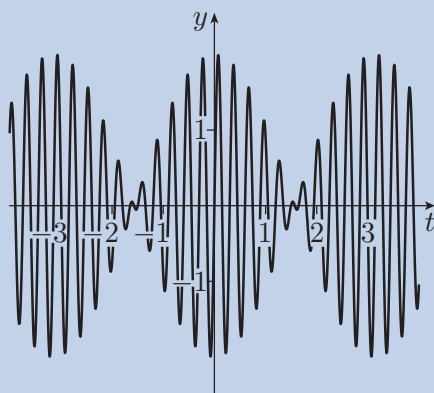


Figure 67 The graph of $h(t) = \sin(20t) + \sin(22t)$

So ‘adding’ these two similar waves produces a wave with an interesting shape, consisting of fast oscillations with a slow variation in the magnitude of these oscillations. This shape can be explained by applying the ‘sum of sines’ identity above to give

$$h(t) = \sin(20t) + \sin(22t) = 2 \sin(21t) \cos t.$$

So the graph of the function h can be thought of as the rapidly oscillating graph of $\sin(21t)$ scaled by the factor $2 \cos t$, which produces a slow variation in magnitude, known as *beating*.

You can observe this phenomenon when two musical notes with similar frequencies are played simultaneously, and the resulting slow variation in magnitude is heard as ‘beats’.

5 Using the computer for trigonometry

Activity 37 Working with trigonometric functions on the computer

To complete your study of Unit 4, work through Section 6 of the *Computer algebra guide*, where you will learn how to use the computer to work with trigonometric functions.



Learning outcomes

After studying this unit, you should be able to:

- use radians as a measure of angle, and convert between radians and degrees
- use sine, cosine and tangent to find unknown side lengths and angles of right-angled triangles
- define the sine, cosine and tangent of any angle
- sketch the graphs of sine, cosine and tangent
- use the ASTC diagram or graphs to find values of sine, cosine and tangent
- define inverse sine, inverse cosine and inverse tangent for any angle
- solve trigonometric equations using the ASTC diagram or graphs
- find unknown side lengths and angles in triangles using the sine rule and cosine rule
- calculate the area of a triangle from two side lengths and the angle between them
- define the cosecant, secant and cotangent of any angle
- understand and use trigonometric identities.

Solutions to activities

Solution to Activity 1

- (a) (i) Applying the degrees-to-radians conversion formula gives

$$1^\circ = \left(\frac{\pi}{180} \times 1 \right) \text{ radians} = \frac{\pi}{180} \text{ radians.}$$

- (ii) Applying the degrees-to-radians conversion formula gives

$$345^\circ = \left(\frac{\pi}{180} \times 345 \right) \text{ radians} = \frac{23\pi}{12} \text{ radians.}$$

- (b) (i) Applying the radians-to-degrees conversion formula gives

$$\frac{3\pi}{5} \text{ radians} = \left(\frac{180}{\pi} \times \frac{3\pi}{5} \right)^\circ = 108^\circ.$$

- (ii) Applying the radians-to-degrees conversion formula gives

$$\frac{7\pi}{4} \text{ radians} = \left(\frac{180}{\pi} \times \frac{7\pi}{4} \right)^\circ = 315^\circ.$$

Solution to Activity 2

- (a) There are 2π radians in a full turn. A sector of $\pi/3$ radians has been removed, and $2\pi - \pi/3 = 5\pi/3$, so the shape is a sector of angle $5\pi/3$ radians.

- (b) The perimeter of the shape consists of two straight lines of length 1 cm, and an arc that subtends an angle of $5\pi/3$ radians. Using the arc length formula gives $5\pi/3 \times 1 = 5\pi/3$, so the arc length is $5\pi/3$ cm. Then

$$1 + 1 + \frac{5\pi}{3} = 2 + \frac{5\pi}{3} = 7.235\dots,$$

so the total perimeter of the shape is 7.2 cm (to 1 d.p.).

- (c) The area of a sector formula gives

$$\frac{1}{2} \times 1^2 \times \frac{5\pi}{3} = \frac{5\pi}{6} = 2.617\dots,$$

so the shape has area 2.6 cm^2 (to 1 d.p.).

Solution to Activity 3

- (a) Pythagoras' theorem says that the square of the hypotenuse is the sum of the squares of the other two sides. The sum of the squares of the other two sides is

$$8^2 + 15^2 = 289.$$

Therefore the length of the hypotenuse is $\sqrt{289} = 17$.

- (b) For this angle θ , opp = 15, adj = 8 and hyp = 17. Therefore

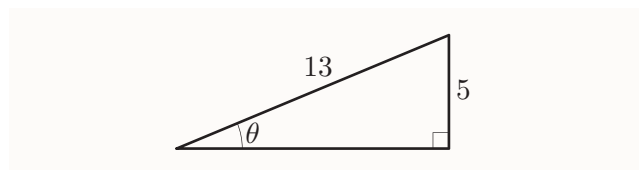
$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{15}{17},$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{8}{17},$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{15}{8}.$$

Solution to Activity 4

Since $\sin \theta = \frac{5}{13}$, you can draw a right-angled triangle with one acute angle θ , such that opp = 5 and hyp = 13.



Then, by Pythagoras' theorem,

$$13^2 = 5^2 + \text{adj}^2,$$

so

$$\text{adj} = \sqrt{169 - 25} = 12.$$

Hence

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13},$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}.$$

Solution to Activity 5

$$\sin 62^\circ = 0.883 \quad (\text{to 3 s.f.})$$

$$\cos 62^\circ = 0.469 \quad (\text{to 3 s.f.})$$

$$\tan 62^\circ = 1.88 \quad (\text{to 3 s.f.})$$

Solution to Activity 6

- (a) The labelled side lengths are opp = 5 and hyp = p , so

$$\sin 35^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{5}{p}.$$

Rearranging this in terms of p gives

$$p = \frac{5}{\sin 35^\circ} = 8.717\dots = 8.7 \quad (\text{to 2 s.f.}).$$

Unit 4 Trigonometry

- (b) The labelled side lengths are $\text{adj} = q$ and $\text{hyp} = 7$, so

$$\cos \frac{\pi}{4} = \frac{\text{adj}}{\text{hyp}} = \frac{q}{7}.$$

Rearranging this in terms of q gives

$$q = 7 \cos \frac{\pi}{4} = 4.949 \dots = 4.9 \text{ (to 2 s.f.)}.$$

- (c) The labelled side lengths are $\text{opp} = r$ and $\text{adj} = 6$, so

$$\tan \frac{\pi}{6} = \frac{\text{opp}}{\text{adj}} = \frac{r}{6}.$$

Rearranging this in terms of r gives

$$r = 6 \tan \frac{\pi}{6} = 3.464 \dots = 3.5 \text{ (to 2 s.f.)}.$$

Solution to Activity 7

By the equation stated before the activity, the width of the river in metres is given by

$$50 \times \tan 57^\circ = 76.993 \dots$$

So the width of the river is 77 m, to the nearest metre.

Solution to Activity 8

Let h be the height of the building in metres. Then

$$\frac{h}{100} = \tan 72^\circ.$$

Therefore

$$h = 100 \tan 72^\circ = 307.768 \dots$$

So the height of the building is 308 m, to the nearest metre.

Solution to Activity 9

- (a) For the angle θ , the side lengths are $\text{opp} = 40$ and $\text{adj} = 35$. Therefore

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{40}{35}.$$

Hence

$$\theta = \tan^{-1} \left(\frac{40}{35} \right) = 48.814 \dots^\circ = 49^\circ,$$

to the nearest degree.

- (b) For the angle α , the side lengths are $\text{opp} = 8$ and $\text{hyp} = 15$. Therefore

$$\sin \alpha = \frac{\text{opp}}{\text{hyp}} = \frac{8}{15}.$$

Hence

$$\alpha = \sin^{-1} \left(\frac{8}{15} \right) = 32.230 \dots^\circ = 32^\circ,$$

to the nearest degree.

For the angle β , the side lengths are $\text{adj} = 8$ and $\text{hyp} = 15$. Therefore

$$\cos \beta = \frac{\text{adj}}{\text{hyp}} = \frac{8}{15}.$$

Hence

$$\beta = \cos^{-1} \left(\frac{8}{15} \right) = 57.769 \dots^\circ = 58^\circ,$$

to the nearest degree.

Alternatively, because the sum of the two acute angles in a right-angled triangle is 90° ,

$$\beta = 90^\circ - \alpha = 57.769 \dots^\circ = 58^\circ,$$

to the nearest degree.

Solution to Activity 10

The identity $\sin^2 \theta + \cos^2 \theta = 1$ gives

$$\sin^2 \theta = 1 - \cos^2 \theta,$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta}.$$

Here the positive square root has been taken because $\sin \theta$ is positive. So

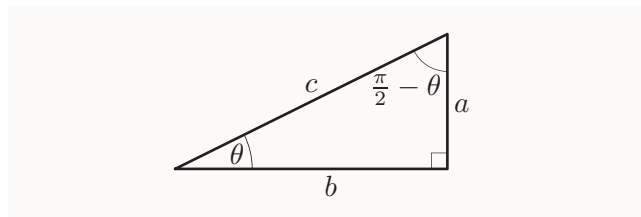
$$\begin{aligned} \sin \theta &= \sqrt{1 - \left(\frac{3}{7} \right)^2} \\ &= \frac{\sqrt{40}}{7} \\ &= \frac{2\sqrt{10}}{7}. \end{aligned}$$

Then

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{2\sqrt{10}/7}{3/7} \\ &= \frac{2\sqrt{10}}{3}. \end{aligned}$$

Solution to Activity 11

Consider the right-angled triangle below (which is the same as the right-angled triangle in Figure 22).



You can see that

$$\tan \theta = \frac{a}{b}, \quad \text{so} \quad \frac{1}{\tan \theta} = \frac{b}{a}.$$

Also,

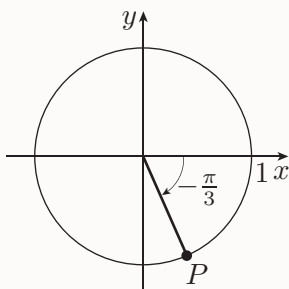
$$\tan \left(\frac{\pi}{2} - \theta \right) = \frac{b}{a}.$$

Therefore

$$\tan \left(\frac{\pi}{2} - \theta \right) = \frac{1}{\tan \theta}.$$

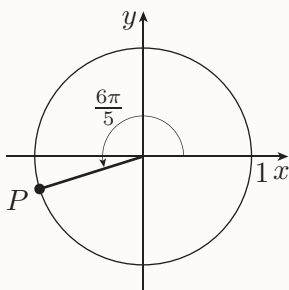
Solution to Activity 12

(a)



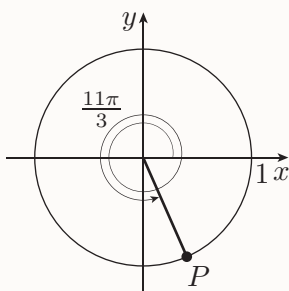
The point P lies in the fourth quadrant.

(b)



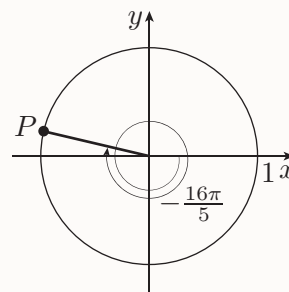
The point P lies in the third quadrant.

(c)



The point P lies in the fourth quadrant.

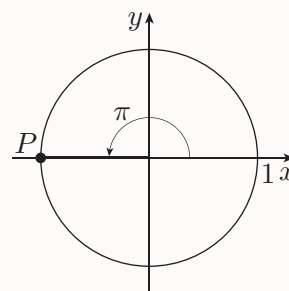
(d)



The point P lies in the second quadrant.

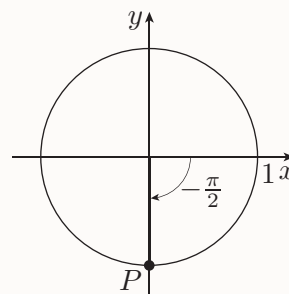
Solution to Activity 13

(a)



The point P has coordinates $(-1, 0)$. Therefore
 $\sin \pi = 0$, $\cos \pi = -1$ and $\tan \pi = 0$.

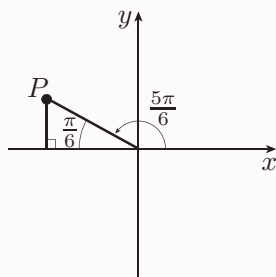
(b)



The point P has coordinates $(0, -1)$. Therefore
 $\sin \left(-\frac{\pi}{2} \right) = -1$, $\cos \left(-\frac{\pi}{2} \right) = 0$
 and $\tan \left(-\frac{\pi}{2} \right)$ is undefined.

Solution to Activity 14

(a)



For the right-angled triangle in the diagram,

$$\sin \frac{\pi}{6} = \frac{\text{opp}}{1} = \text{opp} \quad \text{and} \quad \cos \frac{\pi}{6} = \frac{\text{adj}}{1} = \text{adj}.$$

Since

$$\sin \frac{\pi}{6} = \frac{1}{2} \quad \text{and} \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

the coordinates of P are $(-\sqrt{3}/2, 1/2)$.

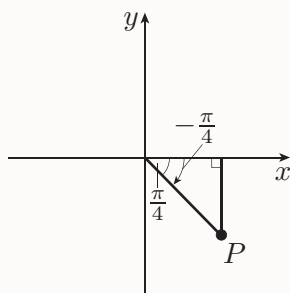
Therefore

$$\sin \left(\frac{5\pi}{6} \right) = \frac{1}{2},$$

$$\cos \left(\frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2},$$

$$\tan \left(\frac{5\pi}{6} \right) = \left(\frac{1}{2} \right) / \left(-\frac{\sqrt{3}}{2} \right) = -\frac{1}{\sqrt{3}}.$$

(b)



For the right-angled triangle in the diagram,

$$\sin \frac{\pi}{4} = \frac{\text{opp}}{1} = \text{opp} \quad \text{and} \quad \cos \frac{\pi}{4} = \frac{\text{adj}}{1} = \text{adj}.$$

Since

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

the coordinates of P are $(1/\sqrt{2}, -1/\sqrt{2})$.

Therefore

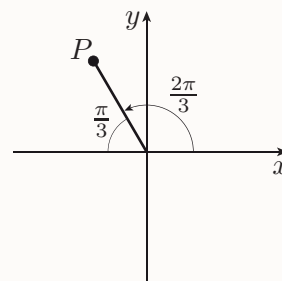
$$\sin \left(-\frac{\pi}{4} \right) = -\frac{1}{\sqrt{2}},$$

$$\cos \left(-\frac{\pi}{4} \right) = \frac{1}{\sqrt{2}},$$

$$\tan \left(-\frac{\pi}{4} \right) = \left(-\frac{1}{\sqrt{2}} \right) / \left(\frac{1}{\sqrt{2}} \right) = -1.$$

Solution to Activity 15

(a)



Since $2\pi/3$ is a second quadrant angle, its sine is positive and its cosine and tangent are negative.

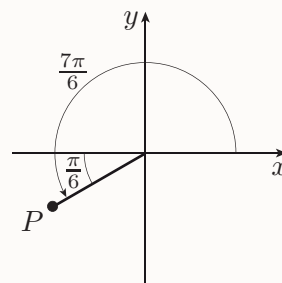
Hence

$$\sin \left(\frac{2\pi}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

$$\cos \left(\frac{2\pi}{3} \right) = -\cos \frac{\pi}{3} = -\frac{1}{2},$$

$$\tan \left(\frac{2\pi}{3} \right) = -\tan \frac{\pi}{3} = -\sqrt{3}.$$

(b)



Since $7\pi/6$ is a third quadrant angle, its tangent is positive and its sine and cosine are negative.

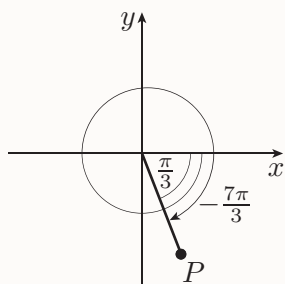
Hence

$$\sin\left(\frac{7\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2},$$

$$\cos\left(\frac{7\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2},$$

$$\tan\left(\frac{7\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

(c)



Since $-7\pi/3$ is a fourth quadrant angle, its cosine is positive and its sine and tangent are negative. Hence

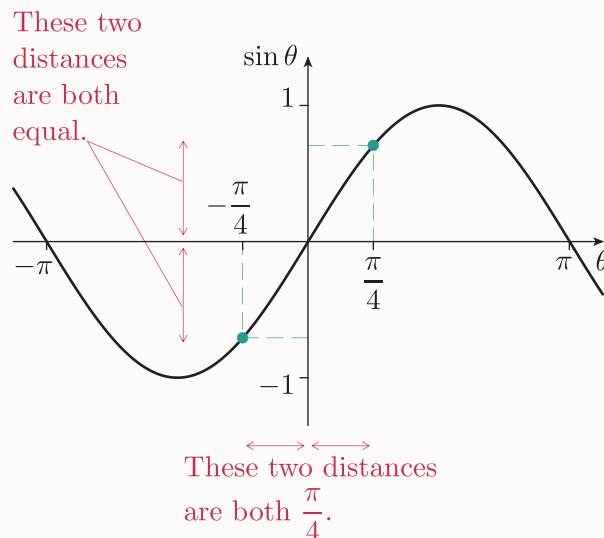
$$\sin\left(-\frac{7\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2},$$

$$\cos\left(-\frac{7\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2},$$

$$\tan\left(-\frac{7\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

Solution to Activity 17

(a)



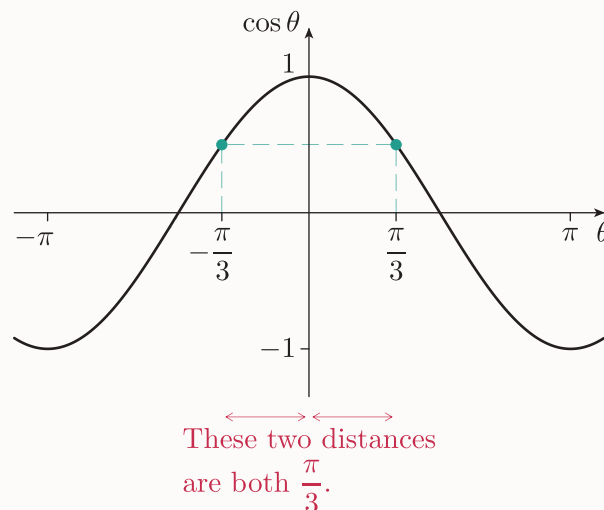
From the graph,

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4}.$$

Since $\sin(\pi/4) = 1/\sqrt{2}$, it follows that

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

(b)



From the graph,

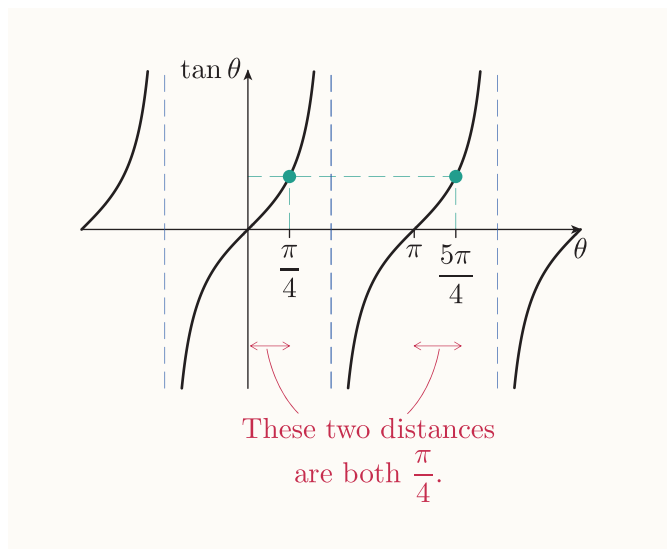
$$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3}.$$

Unit 4 Trigonometry

Since $\cos(\pi/3) = \frac{1}{2}$, it follows that

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}.$$

(c)



From the graph,

$$\tan\left(\frac{5\pi}{4}\right) = \tan\frac{\pi}{4}.$$

Since $\tan(\pi/4) = 1$, it follows that

$$\tan\left(\frac{5\pi}{4}\right) = 1.$$

Solution to Activity 18

(a) $\sin^{-1}(\sqrt{3}/2) = \pi/3$

(b) $\cos^{-1}\frac{1}{2} = \pi/3$

(c) $\cos^{-1}0 = \pi/2$

(d) $\cos^{-1}1 = 0$

(e) $\cos^{-1}(\sqrt{3}/2) = \pi/6$

(f) $\tan^{-1}0 = 0$

(g) $\tan^{-1}1 = \pi/4$

Solution to Activity 19

(a) $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$

(b) $\cos^{-1}(-\frac{1}{2}) = \pi - \cos^{-1}\frac{1}{2} = \pi - \pi/3 = 2\pi/3$

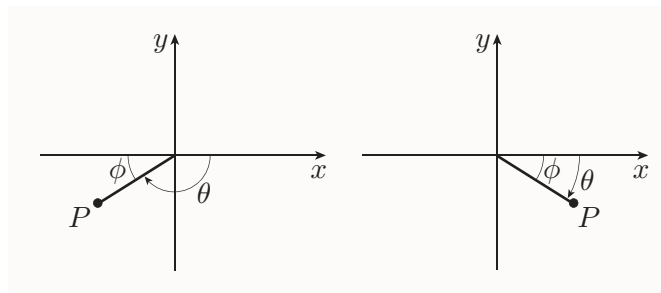
(c) $\cos^{-1}(-1) = \pi - \cos^{-1}1 = \pi - 0 = \pi$

(d) $\cos^{-1}(-\sqrt{3}/2) = \pi - \cos^{-1}(\sqrt{3}/2)$
 $= \pi - \pi/6 = 5\pi/6$

(e) $\tan^{-1}(-1) = -\tan^{-1}1 = -\pi/4$

Solution to Activity 20

- (a) The sine of θ is negative, so θ is a third- or fourth-quadrant angle.



Here

$$\sin\theta = -\frac{1}{2}, \quad \text{so} \quad \sin\phi = \frac{1}{2}.$$

Therefore, from the special angles table,

$$\phi = \pi/6.$$

Hence the solutions of this equation between $-\pi$ and π are

$$\theta = -\pi + \phi = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

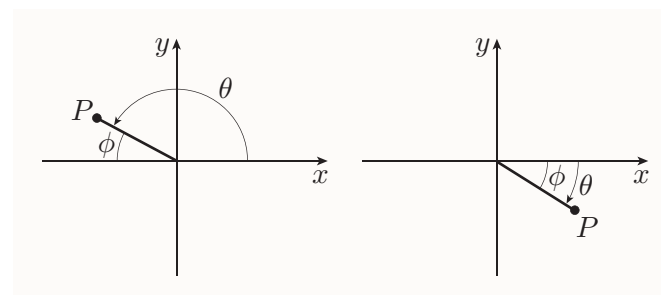
and

$$\theta = -\phi = -\pi/6.$$

Since the sine function has period 2π , the solutions of this equation between π and 3π are

$$-\frac{5\pi}{6} + 2\pi = \frac{7\pi}{6} \quad \text{and} \quad -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}.$$

- (b) The tangent of θ is negative, so θ is a second- or fourth-quadrant angle.



Here

$$\tan\theta = -5, \quad \text{so} \quad \tan\phi = 5.$$

Therefore, using a calculator,

$$\phi = \tan^{-1}5 = 78.690\dots^\circ.$$

Hence

$$\theta = 180^\circ - \phi = 180^\circ - 78.690\dots^\circ = 101.309\dots^\circ$$

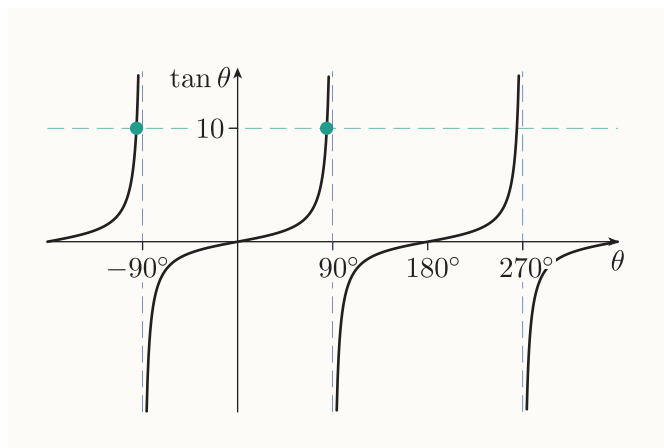
and

$$\theta = -\phi = -78.690\dots^\circ.$$

So the solutions are -79° and 101° , to the nearest degree.

Solution to Activity 21

(a)



The solution between 0° and 90° is

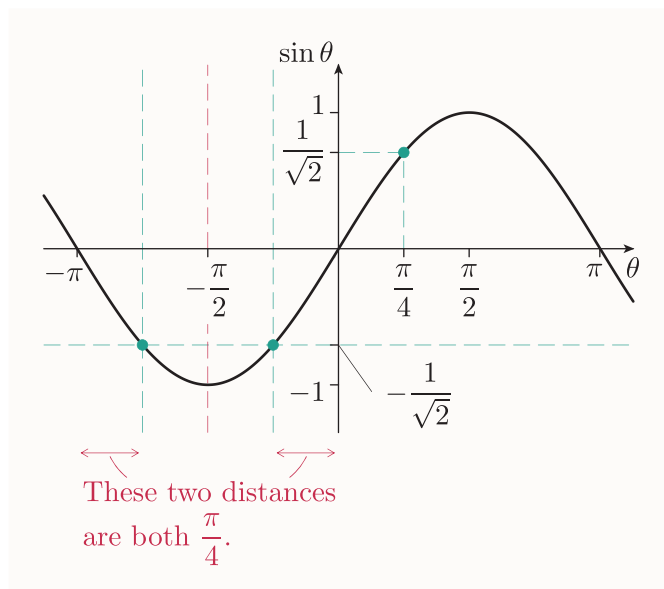
$$\theta = \tan^{-1} 10 = 84.289 \dots^\circ.$$

From the graph, the other solution is

$$\theta = 84.289 \dots^\circ - 180^\circ = -95.710 \dots^\circ.$$

So the solutions are 84° and -96° , to the nearest degree.

(b)



From the graph, one solution is $-\pi/4$.

Symmetry of the graph about the line $x = -\pi/2$ shows that the other solution is

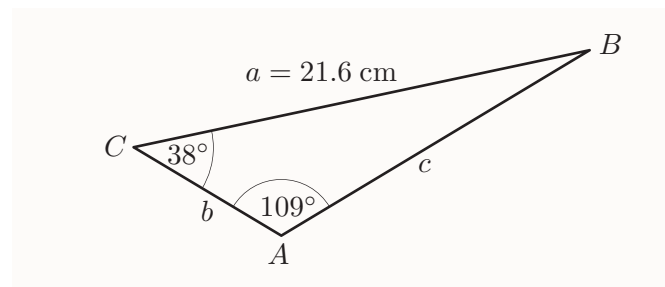
$$\theta = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}.$$

It follows that the solutions are $-\pi/4$ and $-3\pi/4$.

Solution to Activity 22

- (a) The equation $\sin \theta = 0$ tells you that the associated point P on the unit circle with angle θ has y -coordinate 0. That is, P lies on the x -axis, so the solutions in $[-\pi, \pi]$ are $-\pi$, 0 and π .
- (b) The equation $\cos \theta = -1$ tells you that the associated point P on the unit circle with angle θ has x -coordinate -1 . That is, P lies on the negative x -axis, so the solutions in $[0, 4\pi]$ are π and 3π .

Solution to Activity 23



Rearranging the sine rule

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

gives

$$c = \frac{a \sin C}{\sin A} = \frac{21.6 \times \sin 38^\circ}{\sin 109^\circ} = 14.064 \dots$$

Therefore $c = 14.1$ cm, to one decimal place.

Next, the angle B satisfies

$$B + 38^\circ + 109^\circ = 180^\circ, \quad \text{so} \quad B = 33^\circ.$$

Rearranging the sine rule

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

gives

$$b = \frac{a \sin B}{\sin A} = \frac{21.6 \times \sin 33^\circ}{\sin 109^\circ} = 12.442 \dots$$

Therefore $b = 12.4$ cm, to one decimal place.

Solution to Activity 24

In triangle ABC the angle A is $180^\circ - 47^\circ = 133^\circ$, so angle C is $180^\circ - 133^\circ - 22^\circ = 25^\circ$. In this triangle, the side length opposite the angle C is known, so to find the distance BC , use the sine rule in the form

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Rearranging this gives

$$a = \frac{c \sin A}{\sin C} = \frac{100 \sin 133^\circ}{\sin 25^\circ} = 173.053 \dots$$

So the distance BC is 173 m, to the nearest metre.

Finally the height x satisfies the equation

$$x/BC = \sin B, \text{ so}$$

$$x = BC \sin B = 173.053 \dots \times \sin 22^\circ = 64.82 \dots$$

Hence the height of the building is 65 m to the nearest metre.

Solution to Activity 25

Rearranging the sine rule

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

gives

$$\sin C = \frac{c \sin A}{a} = \frac{16.4 \times \sin 19.9^\circ}{7.6} = 0.734 \dots$$

Therefore either

$$C = \sin^{-1}(0.734 \dots) = 47.265 \dots^\circ,$$

or

$$C = 180^\circ - 47.265 \dots^\circ = 132.734 \dots^\circ.$$

But C is acute, so it is less than 90° . Therefore

$$C = 47.265 \dots^\circ = 47.3^\circ \text{ (to 1 d.p.)}.$$

Solution to Activity 26

Rearranging the sine rule

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

gives

$$\sin C = \frac{c \sin B}{b} = \frac{8.1 \sin 23.4^\circ}{9.3} = 0.345 \dots$$

Therefore

$$C = \sin^{-1}(0.345 \dots) = 20.2^\circ$$

or

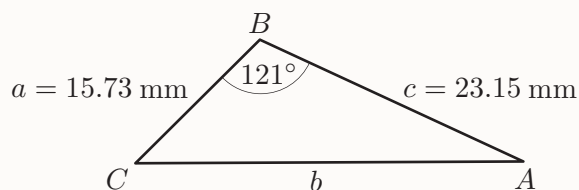
$$C = 180^\circ - 20.2^\circ = 159.8^\circ,$$

to one decimal place. The obtuse angle $C = 159.8^\circ$ is impossible, because

$$159.8^\circ + 23.4^\circ > 180^\circ.$$

Therefore $C = 20.2^\circ$ to one decimal place.

Solution to Activity 27



Applying the cosine rule in the form

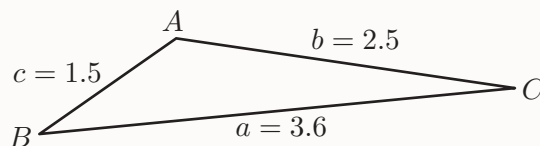
$$b^2 = c^2 + a^2 - 2ca \cos B$$

gives

$$b = \sqrt{23.15^2 + 15.73^2 - 2 \times 23.15 \times 15.73 \cos 121^\circ} = 34.036 \dots$$

Therefore $b = 34.04$ mm, to two decimal places.

Solution to Activity 28



Rearrange the cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$ to give

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

So

$$\cos A = \frac{2.5^2 + 1.5^2 - 3.6^2}{2 \times 2.5 \times 1.5}.$$

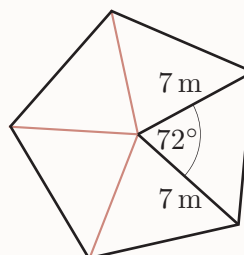
Therefore

$$A = \cos^{-1} \left(\frac{2.5^2 + 1.5^2 - 3.6^2}{2 \times 2.5 \times 1.5} \right) = 126.4 \dots^\circ = 126^\circ,$$

to the nearest degree.

Solution to Activity 29

By drawing lines from the centre of the pentagon to each of the vertices, divide the pentagon into five triangles.



The central angle of each of these triangles is

$$\frac{360^\circ}{5} = 72^\circ.$$

The area of each triangle is

$$\frac{1}{2} \times 7 \times 7 \times \sin 72^\circ = \frac{49}{2} \sin 72^\circ,$$

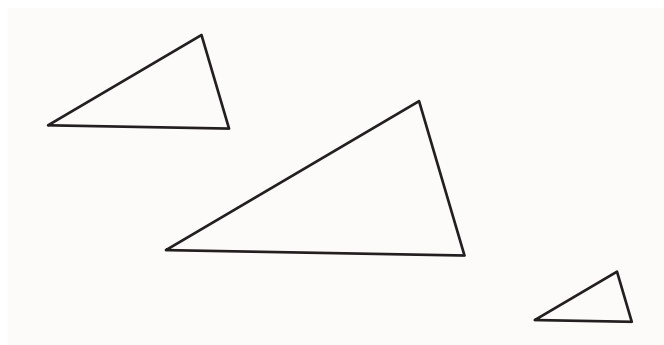
by the formula for the area of a triangle. Then

$$5 \times \frac{49}{2} \sin 72^\circ = 116.504 \dots,$$

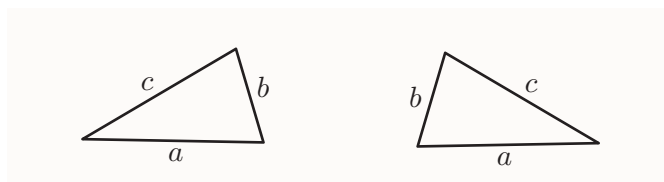
so the area of the pentagon is 116.5 m^2 (to 1 d.p.).

Solution to Activity 30

- (a) No, there is not a formula for area in terms of only A , B and C . This is because similar triangles, which can vary in area, have the same angles. For instance, the three triangles shown below have the same angles but different areas.



- (b) Yes, there is a formula for area in terms of a , b and c . You should expect this, because any two different triangles with side lengths a , b and c are *congruent*, which means that one is a rotated, and possibly flipped over, version of the other. Two congruent triangles (one flipped over) are shown below.



The formula for the area of a triangle in terms of a , b and c is known as *Heron's formula*. It is given by

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$.

Solution to Activity 31

The gradient of the line is $\tan(2\pi/3) = -\sqrt{3}$. The y -intercept is 3. Therefore the equation of the line is $y = -\sqrt{3}x + 3$.

Solution to Activity 32

Rearrange the identity $\sin^2 \theta + \cos^2 \theta = 1$ to give

$$\sin^2 \theta = 1 - \cos^2 \theta.$$

Therefore

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta},$$

and you take the positive square root if $\sin \theta$ is positive, and the negative square root if $\sin \theta$ is negative.

Solution to Activity 33

- (a) Since $\sin 0 = 0$, it follows that $\operatorname{cosec} 0$ and $\cot 0$ are not defined, whereas

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1.$$

- (b) Since $\cos(\pi/2) = 0$, it follows that $\sec(\pi/2)$ is not defined, whereas

$$\operatorname{cosec} \frac{\pi}{2} = \frac{1}{\sin \frac{\pi}{2}} = \frac{1}{1} = 1,$$

and

$$\cot \frac{\pi}{2} = \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} = \frac{0}{1} = 0.$$

- (c) In this case

$$\operatorname{cosec} \frac{\pi}{4} = \frac{1}{\sin \frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2},$$

$$\sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2},$$

$$\cot \frac{\pi}{4} = \frac{\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1.$$

Solution to Activity 34

(a) Let $A = \pi/3$ and $B = \pi/4$, so $A + B = 7\pi/12$.

The identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

gives

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ &= \sin\frac{\pi}{3} \cos\frac{\pi}{4} + \cos\frac{\pi}{3} \sin\frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{1}{4}(\sqrt{6} + \sqrt{2}).\end{aligned}$$

(b) Let $A = \pi/3$ and $B = \pi/4$, so $A - B = \pi/12$.

The identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

gives

$$\begin{aligned}\tan\frac{\pi}{12} &= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3} \tan\frac{\pi}{4}} \\ &= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \\ &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \\ &= \frac{4 - 2\sqrt{3}}{2} \\ &= 2 - \sqrt{3}.\end{aligned}$$

Solution to Activity 35

The half-angle identity for cosine is

$$\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta)).$$

So

$$\begin{aligned}\cos^2 \frac{\pi}{8} &= \frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right) \\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{2} + 1}{2\sqrt{2}} \\ &= \frac{\sqrt{2}(\sqrt{2} + 1)}{2\sqrt{2}\sqrt{2}} \\ &= \frac{2 + \sqrt{2}}{4}.\end{aligned}$$

Because $\pi/8$ is acute, $\cos(\pi/8)$ is positive. Therefore

$$\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}.$$

Solution to Activity 36

The double-angle formula for tangent is

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

So, with $2\theta = \pi/4$,

$$\tan \frac{\pi}{4} = \frac{2 \tan(\pi/8)}{1 - \tan^2(\pi/8)}.$$

Following the hint, put $t = \tan(\pi/8)$. Then, since $\tan(\pi/4) = 1$,

$$\begin{aligned}1 &= \frac{2t}{1 - t^2} \\ 1 - t^2 &= 2t \quad (\text{since } t^2 \neq 1) \\ t^2 + 2t - 1 &= 0.\end{aligned}$$

By the quadratic formula,

$$\begin{aligned}t &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-1)}}{2} \\ &= \frac{-2 \pm \sqrt{8}}{2} \\ &= -1 \pm \sqrt{2}.\end{aligned}$$

Since $t = \tan(\pi/8)$ is positive, the only possibility is

$$\tan \frac{\pi}{8} = \sqrt{2} - 1.$$

(You can check that this answer is the same as the more complicated answer given before the activity, by showing that the squares of both expressions are

equal. Since

$$\left(\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right)^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}}$$

and

$$(\sqrt{2}-1)^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2},$$

you need to check only that

$$\frac{2-\sqrt{2}}{2+\sqrt{2}} = 3 - 2\sqrt{2}.$$

This equation follows by multiplying both sides by $2 + \sqrt{2}$, since

$$(3 - 2\sqrt{2})(2 + \sqrt{2}) = 6 - \sqrt{2} + 4 = 2 - \sqrt{2}.)$$

Acknowledgements

Grateful acknowledgement is made to the following sources:

Figure 6: <http://en.wikipedia.org/wiki/File:Pacman.svg>

Page 27: Maxim Blinkov / www.dreamstime.com

Page 40: Susanne Neal / www.dreamstime.com

Page 68: Peyre, G. and Cohen, L. (2006) 'Geodesic Remeshing Using Front Propagation', International Journal on Computer Vision, Vol.60(1), Springer Publishing

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